

CS720

Logical Foundations of Computer Science

Lecture 18: Simply Typed Lambda Calculus (STLC)

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# What we have learned so far

1. Learned a new programming language called Coq
2. Exercised logic programming and functional programming
3. Formal proof development (inductive definitions, proofs by induction, case analysis)
4. Specify mathematically software systems (ie, formalization process)
5. Designed an imperative programming language (control flow, memory)
6. Proved the correctness of compiler transformations (constant folding)
7. Designed a program verification system (Hoare logic)
8. Proved the correctness of programs (using Hoare logic)
9. Formalized a static analysis (type system)

# Overview: $\lambda$ -calculus

- We want to module the key concept of **functional abstraction**
- We want to represent: function calls, procedures, methods
- We will follow the same pattern:
  1. Language
  2. Small-step semantics
  3. Typing rules
  4. Show Progress and Type Preservation
- Introduce: **variable binding** and **substitution**



# Syntax

```
t ::= x           variable
    | \x:T1.t2    abstraction
    | t1 t2       application
    | true        constant true
    | false       constant false
    | if t1 then t2 else t3  conditional
```

```
T ::= Bool
    | T1 → T2
```

- Our language accepts *higher-order* functions
- All functions are anonymous.
- Only booleans (to simplify presentation)

# Examples

`\x:Bool. x`

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\x:Bool. x
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■ The boolean "not" function.

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(\f:Bool→Bool. f (f true)) (\x:Bool. false)
```

■ The same higher-order function, applied to the constantly false function.

# What is the type of?

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```
┃ Bool→Bool
```

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■ `Bool`

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Booleans: `true`, `false`

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# Values

$$\frac{}{\text{value}(\lambda x : T.t)} \text{ (v-abs)}$$

$$\frac{}{\text{value}(\mathbf{true})} \text{ (v-true)}$$

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Or, formally:

```
Inductive value : tm → Prop :=  
  | v_abs : forall x T t,  
    value (tabs x T t)  
  | v_true :  
    value ttrue  
  | v_false :  
    value tfalse.
```

# Variable substitution

■ We need to represent variable substitution, because of function application.

```
(\ x:Bool. if x then true else x) false
```

Becomes:

```
if false then true else false
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Let us define a **substitution operator**  $[x:=v]$  such that when applied to a term  $t$ , it replaces any occurrence of  $x$  in  $t$  by  $v$ . Care must be taken when handling function abstractions.

In the above example, we would have

```
[x := false] (if x then true else x) = if false then true else false
```

# Substitution examples

`[x:=true] x`

`true`

# Substitution examples

```
[x:=true] x
```

true

```
[x:=true] (if x then x else y)
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# Substitution examples

```
[x:=true] x
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[x:=true] (if x then x else y)
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if true then true else y
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```
[x:=true] y
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# Substitution examples

```
[x:=true] x
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true
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```
[x:=true] (if x then x else y)
```

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if true then true else y
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```
[x:=true] y
```

```
y
```

# Substitution examples

`[x:=true] false`

# Substitution examples

```
[x:=true] false
```

```
false
```

```
[x:=true] (\y:Bool. if y then x else false)
```

# Substitution examples

```
[x:=true] false
```

```
false
```

```
[x:=true] (\y:Bool. if y then x else false)
```

```
\y:Bool. if y then true else false
```

```
[x:=true] (\x:Bool. x)
```

# Substitution examples

```
[x:=true] false
```

```
false
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[x:=true] (\y:Bool. if y then x else false)
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```
[x:=true] (\x:Bool. x)
```

```
\x:Bool. x
```

# Let us implement subst

Formal unit tests:

```
Fixpoint subst (x:string) (s:tm) (t:tm) : tm :=  
(* TODO *)
```

```
Goal subst x ttrue (tvar x) = ttrue. auto. Qed.
```

```
Goal subst x ttrue (tif (tvar x) (tvar x) (tvar y)) = tif ttrue ttrue (tvar y). auto. Qed.
```

```
Goal subst x ttrue (tvar y) = tvar y. auto. Qed.
```

```
Goal subst x ttrue tfalse = tfalse. auto. Qed.
```

```
Goal subst x ttrue (tabs y TBool (tif (tvar y) (tvar x) tfalse)) = tabs y TBool (tif (tvar y)
```

```
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# Variable substitution limitation

What does this produce?

$[ x := y ] (\backslash y:\text{Bool}. x)$

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```
[ x := y ] (\y:Bool. x)
(\y:Bool. y)
```

We consider such a substitution to be ill-formed. That is because  $y$  is a "global" variable defined outside the scope of function  $\lambda y. \text{Bool}. x$ . After substitution, however, variable  $y$  refers to the parameter  $y$  of  $\lambda y. \text{Bool}. x$  which can cause subtle bugs.

In our language, we only care about programs where all variables are introduced via some function abstraction, which negates this situation. Such programs are known as **closed** programs, and these "global" variables, which are not introduced by function abstractions are known as **free** variables.

# Small-step semantics

$$\frac{}{\text{if true then } t_1 \text{ else } t_2 \Rightarrow t_1} \text{(if-true)}$$
$$\frac{}{\text{if false then } t_1 \text{ else } t_2 \Rightarrow t_2} \text{(if-false)}$$
$$\frac{t_1 \Rightarrow t'_1}{\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \Rightarrow \text{if } t'_1 \text{ then } t_2 \text{ else } t_3} \text{(if)}$$

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$$\frac{\text{value}(v_2)}{(\lambda x: T. t_1)v_2 \Rightarrow [x := v_2]t_1} \text{(app-abs)}$$

$$\frac{t_1 \Rightarrow t'_1}{t_1 t_2 \Rightarrow t'_1 t_2} \text{(app1)}$$

$$\frac{\text{value}(v_1) \quad t_2 \Rightarrow t'_2}{v_1 t_2 \Rightarrow v_1 t'_2} \text{(app2)}$$

# Exercise

What is the type of this expression?

```
\x: Bool. y
```

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What is the type of this expression?

```
\x: Bool. y
```

- It depends on the type of  $y$ .
- What is the type of  $y$ ?
- How do we typecheck such an expression?

# Type system

$$\frac{}{\Gamma \vdash \mathbf{true} \in \mathbf{Bool}} \text{(T-true)}$$

$$\frac{}{\Gamma \vdash \mathbf{false} \in \mathbf{Bool}} \text{(T-false)}$$

$$\frac{\Gamma \vdash t_1 \in \mathbf{Bool} \quad \Gamma \vdash t_2 \in T \quad \Gamma \vdash t_3 \in T}{\Gamma \vdash \mathbf{if } t_1 \mathbf{ then } t_2 \mathbf{ else } t_3 \in T} \text{(T-if)}$$

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$$\frac{\Gamma(x) = T}{\Gamma \vdash x \in T} \text{(T-var)}$$

$$\frac{\Gamma \&\{x \mapsto T_p\} \vdash t_b \in T_b}{\Gamma \vdash \lambda x : T_p. t_b \in T_p \rightarrow T_b} \text{(T-abs)}$$

$$\frac{\Gamma \vdash t_f \in T_a \rightarrow T_r \quad \Gamma \vdash t_a \in T_a}{\Gamma \vdash t_f t_a \in T_r} \text{(T-app)}$$

# Summary

- Introduced the Simply-Typed  $\lambda$ -Calculus
- Introduced ***variable binding*** and ***substitution***
- Formalized function (declaration and application) semantics