CS720

Logical Foundations of Computer Science

Lecture 7: Logical connectives in Coq

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What have we learned so far

- Comparing if two expressions are equal syntactically: `e1 = e2`
- Implication `P → Q`
- Universal quantifier `forall x, P`

> Is this all we can do?

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- Comparing if two expressions are equal syntactically: `e1 = e2`
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- Universal quantifier `forall x, P`

> Is this all we can do? No.

We encoded predicates ***computationally***:

- In `Basics.v` we defined `Nat.eqb: nat -> nat -> bool` to compare if two naturals are equal.
- In `Basics.v` we defined `even: nat -> bool` to check if a natural number is even

> Computational predicates are limited in what they can describe (eg, functions in Coq have to be total), and are not very easy to reason about (ie, they are meant to compute/execute, not build logic statements).

Today we will...

- Logical connectives in Coq

$$P \wedge Q \qquad P \vee Q$$

Why are we learning this?

- The building blocks of any interesting property

Typing equality

What is the type of an equality?

```
Check beq_nat 2 2 = true.

Check forall (n m : nat), n + m = m + n.
```

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Typing equality

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Check forall (n m : nat), n + m = m + n.
```

Both of these expressions have type `Prop`, for proposition.

Are all propositions provable?

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Obviously no. How do you prove this proposition:

```
Check 0 = 1. (* Prints: 0 = 1: Prop *)

Goal 0 = 1.
```

Insights

- We can write any proposition, even **unprovable** ones.
  We can write proposition `0 = 1`, but we cannot prove it.
- The fact that something is **false** is not the same as unprovable!
  We ***can prove*** that something is false (by showing it leads to false), eg, `0 = 1`..
  We ***cannot prove*** the law of the excluded middle in Coq.
- In Coq, we must show evidence of what holds. ***(This is known as a constructive logic.)***

Propositions are still expressions (1/3)

What is the type of `ex0`:

```
Definition ex0 := beq_nat 2 2.
```

Propositions are still expressions (1/3)

> What is the type of `ex0`:

```
Definition ex0 := beq_nat 2 2.
```

> What is the type of `ex1`? How can we use `ex1`?

```
Definition ex1 (n:nat) := beq_nat 2 n = true.
Check ex1.
```

> `ex1` is a function that returns a proposition, a parameterized proposition.
> For which `n` is `ex1 n` provable?

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```
Lemma easy:
  forall n, n = 2 -> ex1 n.
Proof.
```

(Done in class.)

Propositions are still expressions (2/3)

> What is the difference between `ex1` and `ex2`?

```
Definition ex1 (n:nat) := beq_nat 2 n = true.

Theorem ex2: forall (n:nat), beq_nat 2 n = true.
```

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Definition ex1 (n:nat) := beq_nat 2 n = true.

Theorem ex2: forall (n:nat), beq_nat 2 n = true.
```

`ex1` defines a position (`Prop`), `ex2` is a theorem definition and is expecting a proof.

> What is the relation between `ex3` and `ex1`, `ex2`?

```
Definition ex3 (n:nat) : beq_nat 2 n = true.
```

Propositions are still expressions (2/3)

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```
Definition ex1 (n:nat) := beq_nat 2 n = true.

Theorem ex2: forall (n:nat), beq_nat 2 n = true.
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`ex1` defines a position (`Prop`), `ex2` is a theorem definition and is expecting a proof.

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```
Definition ex3 (n:nat) : beq_nat 2 n = true.
```

- Recall that `Theorem` and `Definition` are synonyms!
- Thus, `ex2` and `ex3` are the same

Logical connectives

Conjunction

$$P \land Q$$

What is $P \wedge Q$?

1. What is the type of P?

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1. What is the type of P? `Prop`
2. What is the type of Q?

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3. What is the type of $\wedge$?

What is $P \wedge Q$?

1. What is the type of P? `Prop`
2. What is the type of Q? `Prop`
3. What is the type of $\wedge$? `Prop → Prop → Prop`

Split conjunctions in the goal

- When a logical-and appears in the goal, use `split`
- You need to prove both propositions

```
Goal 3 + 4 = 7 /\ 2 * 2 = 4.
Proof.
  split.
```

(Done in class.)

Conjunction example 1

More generally, we can show that if we have propositions A and B, we can conclude that we have $A \wedge B$.

```
Goal forall A B : Prop, A → B → A /\ B.
```

Destruct conjunction in hypothesis

- Case analysis `A /\ B`, how many proofs? how many goals?

```
Goal
  forall x y,
  3 + x = y /\ 2 * 2 = x ->
  x = 4 /\ y = 7.
Proof.
  intros x y Hconj.
  destruct Hconj as [Hleft Hright].
```

(Done in class.)

Conjunction example 2

```
Lemma correct_2 : forall A B : Prop, A /\ B → A.
Proof.

Lemma correct_3 : forall A B : Prop, A /\ B → B.
Proof.
```

(Done in class.)

Disjunction

$$P \vee Q$$

What is $P \vee Q$?

1. What is the type of P?

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1. What is the type of P? `Prop`
2. What is the type of Q?

What is $P \lor Q$?

1. What is the type of P? `Prop`
2. What is the type of Q? `Prop`
3. What is the type of $\lor$?

What is $P \lor Q$?

1. What is the type of P? `Prop`
2. What is the type of Q? `Prop`
3. What is the type of $\lor$? `Prop → Prop → Prop`

Choose disjunction in goal

- Use `left`/`right` to pick what you want to prove
- **Only choose when you know you *can* prove it**

```
Goal
  forall n m : nat, Nat.beq n m = true \/ Nat.beq n m = false.
Proof.
```

Destruct disjunction in hypothesis

- Case analysis `A \/ B`, how many proofs? how many goals?

```
Lemma or_example :
  forall n m : nat, n = 0 \/ m = 0 -> n * m = 0.
Proof.
  intros n m Hor.
  destruct Hor as [Heq | Heq].
```

Falsehood

$$\perp$$

Find contradiction, false in goal

- `False` cannot be proved (we postpone how to our next lecture)
- Equality contradictions can be handled via explosion principle (`discriminate`)
- In this example we show that `1 = 2` is (leads to) false.

```
Goal
  1 = 2 →
  False.
```

Destruct false in hypothesis

- Case analysis concludes any proof with `False` as assumption

```
Theorem ex_falso_quodlibet : forall (P:Prop),
  False → P.
Proof.
```

Negation

$$\neg P$$

Not in assumption and contradictions

```
Definition not (P:Prop) := P → False.
Notation "~ x" := (not x) : type_scope.
```

- apply `~ P` in `P` to reach contradiction
- alternatively, use `contradiction`

```
Theorem contradiction_implies_anything : forall P Q : Prop,
  (P /\ ~ P) → Q.
Proof.
```

Negation in goal, proof by contradiction

- Show `~ P` by assuming `P` and reaching contradiction

```
Goal
    ~ False.
```