CS450

Structure of Higher Level Languages

Lecture 15: Exercises

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Press arrow keys ← → to change slides.

Solving the homework assignment

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1. **learn what each symbol means for each language:** relate the AST to the data-structures (Exercise 1, Exercise 2, Exercise 3)
2. **learn how to write environment operations:** using functions `e:env-get` and `e:env-put`. (Exercise 3)
3. **learn how to implement a function defined by branches:** understand pattern matching and any extra conditions (Exercise 1)
4. **learn how to implement inductive definitions:** reading "fraction" rules; implementing constraints (Exercise 2, Exercise 3)
5. **formal methods:** discussed in the first lecture of this module; investigate and cite your sources (Exercise 4 and 5) — not covered by this lecture

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(1) Learn what each symbol means for each language

Learn what each symbol means for each language

- The homework assignment shows **2 different languages!**
- λ_S: Exercise 1 and Exercise 2
- λ_E: Exercise 3

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λ_S (Exercises 1 and 2)

$$e ::= v \mid x \mid (e_1\ e_2) \qquad v ::= n \mid \lambda x.e$$

```
(define-type s:value (U s:number s:lambda))
(define-type s:expression (U s:value s:variable s:apply))

(struct s:number ([value : Number]))
(struct s:variable ([name : Symbol]))
(struct s:lambda ([param : s:variable] [body : s:expression]))
(struct s:apply ([func : s:expression] [arg : s:expression]))
```

λ_E: values (Exercise 3)

$$v ::= n \mid \{E, \lambda x.e\}$$

```
(define-type e:value (U e:number e:closure))
; n
(struct e:number ([value : Number]))
; {E,λx.e}
(struct e:closure (
    [env : e:environ]      ; E
    [param : e:variable]   ; x
    [body : e:expression]  ; e
  )
)
```

λ_E: expressions (Exercise 3)

$$e ::= v \mid x \mid (e_1\ e_2) \mid \lambda x.e$$

```
(define-type e:expression (U e:value e:variable e:apply e:lambda))
; x
(struct e:variable ([ name : Symbol ]))
; (e1 e2)
(struct e:apply (
    [func : e:expression] ; e1
    [arg : e:expression]  ; e2
))
; λx.e
(struct e:lambda (
    [param : e:variable]   ; x
    [body : e:expression]  ; e
))
```

(2) Learn how to write environment operations

Encoding environments with hash-tables

- How can we encode an empty environment $\emptyset$:

Encoding environments with hash-tables

- How can we encode an empty environment $\emptyset$: (hash)

- How can we encode a lookup $E(x)$:

Encoding environments with hash-tables

- How can we encode an empty environment $\emptyset$: `(hash)`

- How can we encode a lookup $E(x)$: `(e:env-get E x)`

- How can we encode environment extension $E[x \mapsto v]$:

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Encoding environments with hash-tables

- How can we encode an empty environment $\emptyset$: `(hash)`
- How can we encode a lookup $E(x)$: `(e:env-get E x)`
- How can we encode environment extension $E[x \mapsto v]$: `(e:env-put E x v)`

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Test-cases

Test-cases

Function `(check-e:eval? env exp val)` is given in the template to help you test effectively your code.

> The use of `check-e:eval` is **optional**. You are encouraged to play around with `e:eval` directly.

1. The first parameter is an S-expression that represents an **environment**. The S-expression must be a list of pairs representing each variable binding. The keys must be symbols, the values must be serialized λ_E values

```
[] ; The empty environment
[ (x . 1) ]  ; An environment where x is bound to 1
[ (x . 1) (y . 2) ] ; An environment where x is bound to 1 and y is bound to 2
```

2. The second parameter is an S-expression that represents the a valid λ_E **expression**

3. The third parameter is an S-expression that represents a valid λ_E **value**

Serialized expressions in λ_E

> Each line represents a **quoted** expression as a parameter of function `e:parse-ast`. For instance, `(e:parse-ast '(x y))` should return `(e:apply (e:variable 'x) (list (e:variable 'y)))`.

```
1                               ; (e:number 1)
x                               ; (e:variable 'x)
(closure [(y . 20)] (lambda (x) x))
; (e:closure
;       (hash (e:variable 'y) (e:number 20))
;       (e:lambda (list (e:variable 'x)) (list (r:variable 'x))))
(lambda (x) x)                  ; (e:lambda (list (e:variable 'x)) (list (e:variable 'x)))
(x y)                           ; (e:apply (e:variable 'x) (list (e:variable 'y)))
```

Test cases

```
; x is bound to 1, so x evaluates to 1
(check-e:eval? '[(x . 1)] 'x 1)
; 20 evaluates to 20
(check-e:eval? '[(x . 2)] 20 20)
; a function declaration evaluates to a closure
(check-e:eval? '[] '(lambda (x) x) '(closure [] (lambda (x) x)))
; a function declaration evaluates to a closure; notice the environment change
(check-e:eval? '[(y . 3)] '(lambda (x) x) '(closure [(y . 3)] (lambda (x) x)))
; because we use an S-expression we can use brackets, curly braces, or parenthesis
(check-e:eval? '{(y . 3)} '(lambda (x) x) '(closure [(y . 3)] (lambda (x) x)))
; evaluate function application
(check-e:eval? '{} '((lambda (x) x) 3)  3)
; evaluate function application that returns a closure
(check-e:eval? '{} '((lambda (x) (lambda (y) x)) 3)  '(closure {[x . 3]} (lambda (y) x)))
```

Implementing inductive definitions

A primer

Implementing inductive definitions

A primer

> Disciplining an ambiguous presentation medium to communicate a precise mathematical meaning (**notation** and **convention**)

- Implementing algorithms written in a mathematical notation
- Discuss recursive functions (known as inductive definitions)
- Present various design choices
- We are restricting ourselves to the specification of functions
  (If $M(x) = y$ and $M(x) = z$, then $y = z$)

Equation notation

- Function $M(n)$ has one input n and one output after the equals sign.
- Each rule declares some pre-conditions
- The result of the function is only returned if the pre-conditions are met

Formally

$$M(n) = n - 10 \quad \text{if } n > 100$$
$$M(n) = M(M(n + 11)) \quad \text{if } n \leq 100$$

Implementation

- Each branch of the `cond` represents a rule
- The condition of each branch should be the pre-condition

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Implementation

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- The condition of each branch should be the pre-condition

```
(define (M n)
  (cond
    [(> n 100) (- n 10)]
    [(<= n 100) (M (M (+ n 11)))]))
```

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Fraction notation

- We can use the "fraction"-based notation to represent pre-conditions
- Above is a pre-condition, below is the result of the function
- The result is only available if the pre-condition holds

Formally

$$\frac{n > 100}{M(n) = n - 10} \qquad \frac{n \leq 100}{M(n) = M(M(n + 11))}$$

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Multiple pre-conditions in fraction-notation

- Fraction-based notation admits multiple pre-conditions
- The result only happens if **all** pre-conditions are met (logical conjunction)
- We are only interested in function calls that do always succeed (ignore errors)
- Since we are defining functions, only one output is possible at any time

$$\frac{n > 100}{M(n) = n - 10} \qquad \frac{M(n + 11) = x \qquad M(x) = y \qquad n \leq 100}{M(n) = y}$$

- In the second rule, note the implicit dependency between variables
- The dependency between variables, specifies the implementation order (eg, **x** must be defined before **y**)

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```
(define (M n)
  (cond
    [(> n 100) (- n 10)]
    [(<= n 100)
      (define x (M (+ n 11)))
      (define y (M x))
      y]))
```

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The equal sign is optional

- The distinction between input and output should be made clear by the author of the formalism

$$\frac{n > 100}{M(n) = n - 10} \qquad \frac{M(n + 11) = x \qquad M(x) = y \qquad n \leq 100}{M(n) = y}$$

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$$\frac{n > 100}{M(n) = n - 10} \qquad \frac{M(n + 11) = x \qquad M(x) = y \qquad n \le 100}{M(n) = y}$$

We can use any symbol!

Let us define the M function with the 📻 symbol. The intent of notation is to aid the reader and reduce verbosity.

$$\frac{n > 100}{n \,📻\, n - 10} \qquad \frac{n + 11 \,📻\, x \qquad x \,📻\, y \qquad n \le 100}{n \,📻\, y}$$

How do we write $M(M(n + 11))$?

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Pattern matching rules

- The pre-condition is implicitly defined according to the **structure** of the input
- *First rule:* can only be applied if the list is empty
- *Second rule:* can only be applied if there is at least one element in the list

$$qs([]) = []$$

$$\frac{qs([x \mid x < p \wedge x \in l]) = l_1 \qquad qs([x \mid x \geq p \wedge x \in l]) = l_2}{qs(p :: l) = l_1 \cdot [p] \cdot l_2}$$

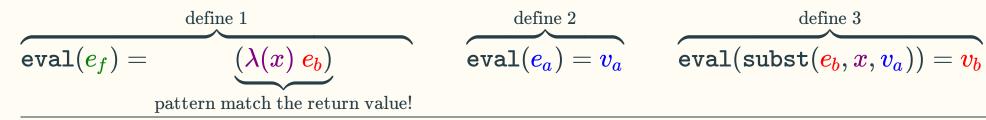UMass
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Pattern matching rules (implementation)

```
(define (qs l)
   (cond [(empty? l) empty] ; qs([]) = []
         [else
          ; Input: p :: r
          (define p (first l))
          (define r (rest l))
          ; qs([ x | x < p /\ x \in l]) = l1
          (define l1 (qs (filter (lambda (x) (< x p)) r)))
          ; qs([ x | x >= p /\ x \in l]) = l2
          (define l2 (qs (filter (lambda (x) (>= x p)) r)))
          ; l1 . p . l2
          (append l1 (cons p l2))]))
```

(4) Implementing constraints

(4) Implementing constraints

$$\overbrace{\texttt{eval}(e_f) = \overbrace{(\lambda(x)\ e_b)}^{\text{define 1}}}$$

$$\underbrace{(\lambda(x)\ e_b)}_{\text{pattern match the return value!}}$$

$$\overbrace{\texttt{eval}(e_a) = v_a}^{\text{define 2}} \qquad \overbrace{\texttt{eval}(\texttt{subst}(e_b, x, v_a)) = v_b}^{\text{define 3}}$$

$$\texttt{eval}((e_f\ e_a)) = \underbrace{v_b}_{\text{return this}}$$

Homework assignment

- **Exercise 1.** Function $e[x := v]$ is $(\texttt{s:subst exp var val})$, where e is $\texttt{exp}$, x is $\texttt{var}$, and v is $\texttt{val}$.

- **Exercise 2.** Function $e \Downarrow v$ is $(\texttt{s:eval subst exp})$, where e is $\texttt{exp}$, v is the return value (not displayed in the function signature).

> In the exercise, parameter $\texttt{subst}$ represents the substitution function (local tests use your own implementation, remote tests use a correct implementation of subst).

- **Exercise 3.** Function $e \Downarrow_E v$ is $(\texttt{e:eval env exp})$, where e is $\texttt{exp}$, E is $\texttt{env}$, v is the return value (not displayed in the function signature).

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Church's encoding

Church's encoding

- <u>Alonzo Church</u> created the λ-calculus
- <u>Church's Encoding</u> is a treasure trove of λ-calculus expressions: it shows how natural numbers can be encoded
- Let us go through Church's encoding of booleans
- Examples taken from <u>Colin Kemp's PhD thesis</u> (page 17)

Encoding Booleans with λ-terms

Why? Because you will be needing test-cases.

```racket
(require rackunit)
(define ns (make-base-namespace))
(define (run-bool b) (((eval b ns) #t) #f))

; True
(define TRUE '(lambda (a) (lambda (b) a)))
(define FALSE '(lambda (a) (lambda (b) b)))
(define (OR a b) (list (list a TRUE) b))
(define (AND a b) (list (list a b) FALSE))
(define (NOT a) (list (list a FALSE) TRUE))
(define (EQ a b) (list (list a b) (NOT b)))
; Test
(check-equal?
    (run-bool (EQ TRUE (OR (AND FALSE TRUE) TRUE)))
    (equal? #t (or (and #f #t) #t)))
```

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Hash-tables

TL;DR: A data-structure that stores pairs of key-value entries. There is a lookup operation that given a key retrieves the value associated with that key. Keys are unique in a hash-table, so inserting an entry with the same key, replaces the old value by the new value.

- Hash-tables represent a (partial) <u>injective function</u>.
- Hash-tables were covered in <u>CS310</u>.
- Hash-tables are also known as maps, and dictionaries. We use the term hash-table, because that is how they are known in Racket.

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Hash-tables in Racket

Constructors

1. Function `(hash k1 v1 ... kn vn)` a hash-table with the given key-value entries. Passing zero arguments, `(hash)`, creates an empty hash-table.
2. Function `(hash-set h k v)` copies hash-table `h` and adds/replaces the entry `k v` in the new hash-table.

Accessors

- Function `(hash? h)` returns `#t` if `h` is a hash-table, otherwise it returns `#f`
- Function `(hash-count h)` returns the number of entries stored in hash-table `h`
- Function `(hash-has-key? h k)` returns `#t` if the key is in the hash-table, otherwise it returns `#f`
- Function `(hash-ref h k)` returns the value associated with key `k`, otherwise aborts

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Hash-table example

```
(define h (hash))                    ; creates an empty hash-table
(check-equal? 0 (hash-count h))      ; we can use hash-count to count how many entries
(check-true (hash? h))               ; unsurprisingly the predicate hash? is available

(define h1 (hash-set h "foo" 20))    ; creates a new hash-table where "foo" is bound to 20
(check-equal? (hash "foo" 20) h1)    ; (hash-set (hash) "foo" 20) = (hash "foo" 20)

(define h2 (hash-set h1 "foo" 30))
(check-equal? (hash "foo" 30) h2)    ; in h2 "foo" is the key, and 30 the value
(check-equal? 30 (hash-ref h2 "foo")) ; ensures that hash-ref retrieves the value of "foo"
(check-equal? (hash "foo" 20) h1)    ; h1 remains the same
```

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