CS720

Logical Foundations of Computer Science

Lecture 19: Type systems

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HW8: Equiv. v

Due Tuesday November 6, 11:59pm EST
HW10: Smallstep.v

Due Thursday November 8, 11:59pm EST
HW9: Hoare.v, HoareAsLogic.v, Hoare2.v

Due Friday November 9, 11:59pm EST
HW11: Types.v

Your presentation's title and abstract

Due Thursday November 15, 11:59pm EST
What is a Type System

1. **Asserts that a term is well-formed**: eg, consider a fraction represented by two integers, assert that the denominator is not a zero; eg, all functions terminate;

2. **Asserts that a term is of a given category**: eg, an expression is numeric; eg, a file-pointer is in an open state

How does a Type System work

- Performed at compile time (a static analysis technique)
- Enforces policies to **guarantees certain properties** statically: eg, in Rust, memory is manually allocated, but no memory is leaked, no data-races errors; eg, in Java, the method of a method calls must be known at compile-time and the argument-type must match the parameter-type.
Limitations of IMP

One of the limitations of IMP is that our expressions can only have one type:

- Boolean expressions can only appear in loops/ifs
- Assignments only accept numeric expressions (no booleans)
Introducing data of different types
Let us define an expression language

\[ t ::= \text{true} \mid \text{false} \mid \text{if } t \text{ then } t \text{ else } t \mid 0 \mid \text{succ } t \mid \text{pred } t \mid \text{iszero } t \]

Example:

\[ \text{if iszero (succ (succ(0))) then 0 else pred (succ(succ(0)))} \]

Ill-formed example:

\[ \text{succ(true)} \]
Values

\[
\begin{align*}
\text{bvalue}(\text{true}) & \quad (\text{bv-true}) \\
\text{bvalue}(\text{false}) & \quad (\text{bv-false}) \\
\text{nvalue}(0) & \quad (\text{nv-zero}) \\
\text{nvalue}(\text{succ}(v)) & \quad (\text{nv-succ}) \\
\text{value}(v) & := \text{bvalue}(v) \lor \text{nvalue}(v)
\end{align*}
\]
Value or not a value?

1. succ(if true then succ(0) else 0):
Value or not a value?

1. $\text{succ(if true then succ(0) else 0)}$: Not a value.
2. false:
Value or not a value?

1. succ(if true then succ(0) else 0): Not a value.
2. false: A value.
3. iszero(0):
Value or not a value?

1. succ(if true then succ(0) else 0): Not a value.
2. false: A value.
3. iszero(0): Not a value.
4. succ(0):
Value or not a value?

1. `suc((if true then succ(0) else 0))`: Not a value.
2. `false`: A value.
3. `iszero(0)`: Not a value.
4. `suc(0)`: A value.
Semantics

\[
\frac{\text{true}}{t_1} \quad \text{(IfTrue)}
\]

\[
\frac{\text{false}}{t_2} \quad \text{(IfFalse)}
\]

\[
\frac{t_1 \Rightarrow t_1'}{t_1 \Rightarrow t_2 \Rightarrow t_2'} \quad \text{(If)}
\]

\[
\frac{\text{Succ} \quad \text{(Succ)}}{t_1 \Rightarrow t_1'}
\]

\[
\frac{\text{PredZero} \quad \text{(PredZero)}}{\text{pred}(0) \Rightarrow 0}
\]

\[
\frac{\text{PredSucc} \quad \text{(PredSucc)}}{\text{pred}(\text{succ}(v)) \Rightarrow v}
\]

\[
\frac{\text{PredSucc} \quad \text{(PredSucc)}}{\text{pred}(t) \Rightarrow \text{pred}(t')}
\]

\[
\frac{\text{IszeroZero} \quad \text{(IszeroZero)}}{\text{iszero}(0) \Rightarrow \text{true}}
\]

\[
\frac{\text{IszeroSucc} \quad \text{(IszeroSucc)}}{\text{iszero}(\text{succ}(v)) \Rightarrow \text{false}}
\]

\[
\frac{\text{Iszero} \quad \text{(Iszero)}}{t \Rightarrow t' \Rightarrow \text{iszero}(t')}
\]
Reduction example (1)

if iszero(succ(succ(0))) then 0 else pred(succ(succ(0)))
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if iszero(succ(succ(0))) then 0 else pred(succ(succ(0)))

⇒ If, IszeroSucc
if false then 0 else pred(succ(succ(0)))
Reduction example (1)

if iszero(succ(succ(0))) then 0 else pred(succ(succ(0)))

⇒ If, IszeroSucc
if false then 0 else pred(succ(succ(0)))

⇒ IfFalse
pred(succ(succ(0))))
Reduction example (1)

if iszero(succ(succ(0))) then 0 else pred(succ(succ(0)))

⇒ If, IszeroSucc
if false then 0 else pred(succ(succ(0)))

⇒ IfFalse
pred(succ(succ(0)))

⇒ PredSucc
succ(0)
Reduction example (2)

pred(false)
Reduction example (2)

pred(false)

How do we reduce now?
How do we reduce now?

Some terms are invalid! These are expressions for which we want to consider to be malformed somehow.

Which means our language does not enjoy the process of strong progress.
Stuck terms

Let us define the notion of stuck.

\[
\text{stuck}(t) := \neg \text{value}(t) \land \text{nf}(t)
\]

Think of it as a negation of progress (which says that a term is either a value or reduces)

Example

\[
\frac{\text{nf}(\text{pred(zero)}) \quad \neg \text{value}(\text{pred(zero))}}{\text{stuck}(\text{pred(zero)})}
\]
Example: Stuck or not stuck?

1. \texttt{iszero(if true then succ(0) else 0)}
Example: Stuck or not stuck?

1. \texttt{iszero(if true then succ(0) else 0)} Not stuck.

Is it a value or does it reduce?
Example: Stuck or not stuck?

1. \( \text{iszero}(\text{if true then succ(0) else 0}) \) Not stuck.

Is it a value or does it reduce?

Reduces.

What does it reduce to?
Example: Stuck or not stuck?

1. `iszero(if true then succ(0) else 0)` Not stuck.

   - Is it a value or does it reduce?
   - Reduces.

   - What does it reduce to?

     `iszero(if true then succ(0) else 0) \Rightarrow (IfTrue) iszero(succ(0)) \Rightarrow (IszeroSucc) false`

2. `if succ(0) then true else false`
Example: Stuck or not stuck?

1. \texttt{iszero(if true then succ(0) else 0)} Not stuck.

- Is it a value or does it reduce?
  
  Reduces.

- What does it reduce to?
  
  \texttt{iszero(if true then succ(0) else 0) \rightarrow (IfTrue) iszero(succ(0)) \rightarrow (IszeroSucc) false}

2. \texttt{if succ(0) then true else false} Stuck. Why?
Example: Stuck or not stuck?

1. `iszero(if true then succ(0) else 0)` Not stuck.

   Is it a value or does it reduce?

   Reduces.

   What does it reduce to?

   `iszero(if true then succ(0) else 0) \rightarrow (IfTrue) iszero(succ(0)) \rightarrow (IszeroSucc) false`

2. `if succ(0) then true else false` Stuck. Why? The if expects a boolean.
Type system

- A type system is a set of rules that disciplines expression composition.
- Our expressions can have different types: numerical or boolean
- A type system holds when an expression is of a given type

\[ \vdash t : T \]

In our language our types are:

\[ T ::= \text{Bool} \mid \text{Nat} \]
Defining a Type System (1/2)

Boolean values:

\[ \vdash \text{true} : \text{Bool} \quad \text{(t-true)} \quad \vdash \text{false} : \text{Bool} \quad \text{(t-false)} \]
Defining a Type System (1/2)

Boolean values:

\[ \vdash \text{true} : \text{Bool} \quad (t\text{-true}) \]
\[ \vdash \text{false} : \text{Bool} \quad (t\text{-false}) \]

Natural values:

\[ \vdash 0 : \text{Nat} \quad (t\text{-zero}) \]
\[ \vdash \text{succ}(t) : \text{Nat} \quad (t\text{-succ}) \]
Defining a Type System (1/2)

Boolean values:

\[ \vdash \text{true}: \text{Bool} \quad \vdash \text{false}: \text{Bool} \]

Natural values:

\[ \vdash 0: \text{Nat} \quad \vdash \text{succ}(t): \text{Nat} \]

Composed expressions:

\[ \vdash t: \text{Nat} \quad \vdash \text{iszero}(t): \text{Bool} \]
\[ \vdash t: \text{Nat} \quad \vdash \text{pred}(t): \text{Nat} \]
How do we write the rule for `if`?

\[
\frac{\vdash t_1: ??? \quad \vdash t_2: ??? \quad \vdash t_3: ???}{\vdash \text{if } t_1 \text{ then } t_2 \text{ else } t_3: ???}
\]

(t-if)
Defining a Type System (2/2)

How do we write the rule for if?

\[
\begin{align*}
\vdash t_1 : \text{Bool} & \quad \vdash t_2 : T & \quad \vdash t_3 : T \\
\vdash \text{if } t_1 \text{ then } t_2 \text{ else } t_3 : T
\end{align*}
\] (t-if)

Notice how both branches have the same type!
Example 1:

\[
\begin{align*}
&\vdash 0 : \text{Nat} \\
&\vdash \text{succ}(0) : \text{Nat} \\
&\vdash \text{succ}(\text{succ}(0)) : \text{Nat} \\
&\vdash \text{iszero}(\text{succ}(\text{succ}(0))) : \text{Bool} \\
&\vdash \text{if iszero}(\text{succ}(\text{succ}(0))) \text{ then } 0 \text{ else } \text{pred}(\text{succ}(\text{succ}(0))) : \text{Nat}
\end{align*}
\]
Examples

Example 1:

\[ \begin{align*}
\vdash & 0 : \text{Nat} \\
\vdash & \text{succ}(0) : \text{Nat} \\
\vdash & \text{succ}(\text{succ}(0)) : \text{Nat} \\
\vdash & \text{iszero}(\text{succ}(\text{succ}(0))) : \text{Bool} \\
\vdash & \text{if iszero}(\text{succ}(\text{succ}(0))) \text{ then } 0 \text{ else } \text{pred}(\text{succ}(\text{succ}(0))) : \text{Nat}
\end{align*} \]

Example 2:

\[ \vdash \text{succ}(\text{true}) \]
Progress

Theorem. If \( \vdash t : T \), then \( \text{value}(t) \lor \exists t', t \Rightarrow t' \).

We call this progress and not \textit{strong} progress because this theorem is restricted to well-typed terms.

What is the relation between the \textit{progress} property defined here and the \textit{strong progress} from \textit{SmallStep}?

1. No difference
2. Progress implies strong progress
3. Strong progress implies progress
4. They are unrelated properties
**Progress**

**Theorem.** If $\vdash t : T$, then $\text{value}(t) \lor \exists t', t \Rightarrow t'$.

We call this progress and not *strong* progress because this theorem is restricted to well-typed terms.

What is the relation between the *progress* property defined here and the *strong progress* from SmallStep?

1. No difference
2. Progress implies strong progress
3. Strong progress implies progress
4. They are unrelated properties

**Strong progress implies progress.**
Theorem. If $t : T$, then $\text{value}(t) \lor \exists t', t \Rightarrow t'$.

The proof follows by induction on the derivation of the hypothesis. At each case we have that the simpler term is well typed and that the term is either a value or it reduces.

- In the case that the simpler term is a value, we use the canonical properties, to show that our goal is also a value.
- In the case that the simpler term can reduce, we use apply the reduction rule for the given term to reduce the goal.

```plaintext
Lemma bool_canonical : \forall t, 
    \|- t \in TBool \rightarrow value t \rightarrow bvalue t.

Lemma nat_canonical : \forall t, 
    \|- t \in TNat \rightarrow value t \rightarrow nvalue t.
```
Quiz

Is every well-typed normal form is a value?
Quiz

- Is every well-typed normal form is a value?
  Yes! A corollary of the progress theorem.

- Is every value is a normal form?
Quiz

- Is every well-typed normal form is a value?
  Yes! A corollary of the progress theorem.

- Is every value is a normal form?
  Yes! Homework value_is_nf.

- Is the single-step reduction relation a total function?
Quiz

Is every well-typed normal form is a value?
Yes! A corollary of the progress theorem.

Is every value is a normal form?
Yes! Homework value_is_nf.

Is the single-step reduction relation a total function?
No. Counter-example: reducing a value.
Type preservation

**Theorem.** If \( \vdash t : T \) and \( t \Rightarrow t' \), then \( \vdash t' : T \).

Type preservation establishes the robustness of our type system: a static (compile-time) abstraction is ensured in *all executions* of any accepted program. Otherwise, our type system could say an expression returns a number and upon executing that expression we find out it actually returns a boolean.
Type preservation (proof)

**Theorem.** If $\vdash t : T$ and $t \Rightarrow t'$, then $\vdash t' : T$.

The proof follows by induction on the derivation of the first hypothesis. At each case we must invert the hypothesis that the term reduces. The proof for each case is trivial, as we simply need to apply the typing rule for each term.
Type soundness

Theorem. If $\vdash t : T$ and $t \Rightarrow^* t'$, then $\neg\text{stuck}(t')$.

Type soundness tells us that all well-typed programs never reach a stuck state.
Deterministic step

**Theorem.** If $x \Rightarrow y_1$ and $x \Rightarrow y_2$, then $y_1 = y_2$.

Proof by induction on the derivation of the first hypothesis. At each of the 10 cases, we need to invert the second hypothesis $x \Rightarrow y_2$, which yields 22 cases. Use `auto` and `solve_by_invert` to take care of boring cases (8 cases should remain).

- At cases such as `ST_If` and `ST_Succ` we can simply use the induction hypothesis to rewrite the output term of reducing $t_1$.
- The remaining cases all follow the same structure: they reach a contradiction (remember to use `exfalso`). For instance, in the case for rule `ST_PredSucc`, we have that $t_1$ is a nat-value and that $\text{succ}(t_1) \Rightarrow t'_1$. We conclude by inverting the latter, and using lemma `nvalue_no_step`. 
