Overview

- Introduction of small-step semantics
- Normalization of terms
- Relationship between small-step and big-step semantics.
Revisiting arithmetic semantics

A language with constants and a plus-operator:

\[ t ::= n \mid t \oplus t \]

How do we represent an evaluation function for a term, say \texttt{eval}(t)?
Revisiting arithmetic semantics

A language with constants and a plus-operator:

\[ t ::= n \mid t \oplus t \]

How do we represent an evaluation function for a term, say \( \text{eval}(t) \)?

\[ \text{eval}(n) = n \]

\[ \text{eval}(t_1 \oplus t_2) = \text{eval}(t_2) + \text{eval}(t_2) \]

How do we represent \( \text{eval}(t) \) as a relation \( t \Downarrow n \)?
Revisiting arithmetic semantics

A language with constants and a plus-operator:

\[ t ::= n \mid t \oplus t \]

- How do we represent an evaluation function for a term, say \( \text{eval}(t) \)?
  
  \[ \text{eval}(n) = n \]
  
  \[ \text{eval}(t_1 \oplus t_2) = \text{eval}(t_2) + \text{eval}(t_2) \]

- How do we represent \( \text{eval}(t) \) as a relation \( t \downarrow n \)?

\[
\begin{array}{c}
\text{eval}(n) = n \\
\text{eval}(t_1 \oplus t_2) = \text{eval}(t_2) + \text{eval}(t_2)
\end{array}
\]
Small-step operational semantics

The idea is to model computation similar to how a computer (or an abstract machine) would run. We want to model the smallest step of computation (capture each tick the machine does).

In big step semantics, we are only interested in the outcome of a computation. In small step semantics, we are interested in how the execution unfolds.

A big-step semantics execution can be encoded as a sequence of multiple steps in small-step semantic.

\[
\begin{align*}
(n_1 \oplus n_2) & \Rightarrow n_1 + n_2 \quad \text{(P-const)} \quad \\
(t_1 \oplus t_2) & \Rightarrow t_1' \oplus t_2' \quad \text{(P-left)} \quad \\
(t_2) & \Rightarrow t_2' \quad \text{(P-right)} \\
(n_1 \oplus t_2) & \Rightarrow n_1 \oplus t_2' 
\end{align*}
\]
Example

Step 1:

\[
\begin{align*}
0 \oplus 3 & \Rightarrow 3 \quad \text{(P-const)} \\
(0 \oplus 3) \oplus (2 \oplus 4) & \Rightarrow 3 \oplus (2 \oplus 4) \quad \text{(P-left)}
\end{align*}
\]

Step 2:

\[
\begin{align*}
2 \oplus 4 & \Rightarrow 6 \quad \text{(P-const)} \\
3 \oplus (2 \oplus 4) & \Rightarrow 3 \oplus 6 \quad \text{(P-right)}
\end{align*}
\]

Step 3:

\[
3 \oplus 6 \Rightarrow 9 \quad \text{(P-const)}
\]

We may just write the short-hand notation:

\[
(0 \oplus 3) \oplus (2 \oplus 4) \Rightarrow 3 \oplus (2 \oplus 4) \Rightarrow 3 \oplus 6 \Rightarrow 9
\]
Abstracting a binary relation

Notice how our small-step semantics always only has a unique "output". In such a case, we say that a relation is deterministic. That is, a deterministic relation describes an injective function.

**Definition (deterministic relation).** If $t_1 \Rightarrow t_2$ and $t_1 \Rightarrow t'_2$, then $t_2 = t'_2$. 
Deterministic relations

Theorem. ⇒ is deterministic.

Can you come up with a way to make ⇒ non-deterministic?
Deterministic relations

Theorem. $\Rightarrow$ is deterministic.

Can you come up with a way to make $\Rightarrow$ non-deterministic?

\[
\begin{align*}
(n_1 \oplus n_2) \Rightarrow (n_1 + n_2) & \quad (\text{P-const}) \\
(t_1 \oplus t_2) \Rightarrow (t'_1 \oplus t_2) & \quad (\text{P-left}) \\
(t_2 \Rightarrow t'_2) & \quad (\text{P-right}) \\
(t_1 \oplus t_2) \Rightarrow (t_1 \oplus t'_2)
\end{align*}
\]
Deterministic relations

**Theorem.** ⇒ is deterministic.

Can you come up with a way to make ⇒ non-deterministic?

\[
\begin{align*}
\frac{n_1 \oplus n_2 \Rightarrow n_1 + n_2}{(P\text{-const})} \\
\frac{t_1 \Rightarrow t_1'}{t_1 \oplus t_2 \Rightarrow t_1' \oplus t_2}{(P\text{-left})} \\
\frac{t_2 \Rightarrow t_2'}{t_1 \oplus t_2 \Rightarrow t_1 \oplus t_2'}{(P\text{-right})}
\end{align*}
\]

\[
(0 \oplus 3) \oplus (2 \oplus 4) \Rightarrow 3 \oplus (2 \oplus 4) \\
(0 \oplus 3) \oplus (2 \oplus 4) \Rightarrow (0 \oplus 3) \oplus 6
\]
Small-step semantics as an abstract machine

We can think of the execution of an expression as:

1. The state of the machine is a term $t$
2. A step of the machine performs an atomic unit of computation (e.g., evaluates one addition in a sub-expression)
3. The machine *halts* when it cannot perform any more steps.

Here is an inductive definition of $\text{value}$:

$$\text{value}(n) : (V\text{-nat})$$
Exercise

Which of these are provable?

- $\text{value}(10)$
- $\text{value}(10 \oplus 2)$
- $\exists n, \text{value}(n \oplus 10)$
By convention we write a value (a halted state) as $v$.

\[
\begin{align*}
&n_1 \oplus n_2 \Rightarrow n_1 + n_2 \quad \text{(P-const)} \\
t_1 \oplus t_2 \Rightarrow t_1' \oplus t_2 \quad \text{(P-right)} \\
\text{value}(v_1) \Rightarrow t_2 \Rightarrow t_2' \quad \text{(P-left)} \\
v_1 \oplus t_2 \Rightarrow t_1 \oplus t_2' 
\end{align*}
\]
Strong Progress

- Is our semantics always able to perform a step?
Strong Progress

- Is our semantics always able to perform a step? No. When a term is a value, then there is no rule that we can apply to perform a step.
- If the term is not a value, is it always able to perform a step?
Strong Progress

- Is our semantics always able to perform a step? No. When a term is a value, then there is no rule that we can apply to perform a step.
- If the term is not a value, is it always able to perform a step? Yes.

**Theorem (Strong Progress).** Given a single-step relation ($\Rightarrow$) and a notion of value $\text{value}$. Any term $t$ is either a value $\text{value}(t)$ or it reduces $\exists t', t \Rightarrow t'$.

- A language may not enjoy progress because we "forgot" to write a rule for a given command. Example: extend the grammar to include the minus-operator, but do not update the small-step semantics.
- In concurrency theory, the notion of progress may capture the notion of deadlock freedom (ie, there is always a task that can perform an action, or all tasks are idle). Thus, many concurrent languages do not enjoy progress. (We will cover this Thursday.)
- A language may not reduce because there are type-mismatch errors. (We will cover this later.)
Thinking of the state in terms of steps

If a language enjoys progress, then we can always perform a step until we reach a value.

- Can we perform a step on a value?
- What do we call a term where there are no further steps?
Normal Form

*Normal form.* A term that cannot perform any step (ie, make any progress).

\[
nf(t) := \neg \exists t', t \Rightarrow t'
\]

In our language, are all values in the normal form? Are all normal form terms a value?
Normal Form

*Normal form.* A term that cannot perform any step (i.e., make any progress).

\[ \text{nf}(t) := \neg \exists t', t \Rightarrow t' \]

In our language, are all values in the normal form? Are all normal form terms a value? Yes!

**Theorem.** \( \text{value}(t) \iff \text{nf}(t) \).  

This is a non-trivial result, depending on the language:

- One captures the notion of a halted state **syntactically** (a value)
- The other captures the notion of a halted state **semantically** (in terms of \( \Rightarrow \))
Multi-Step Reduction

Our goal is to relate big-step and small-step semantics.

\[ t \Rightarrow^* t \text{(R-refl)} \quad t_1 \Rightarrow t_2 \Rightarrow^* t_3 \text{(R-step)} \]

- This family of relations is also known as the (reflexive) transitive-closure of a relation.
- The multi-step reduction can be thought of describing all states that can be reached from a given starting state.
Exercise

Recall the following propositions:

$$(0 \oplus 3) \oplus (2 \oplus 4) \Rightarrow 3 \oplus (2 \oplus 4) \Rightarrow 3 \oplus 6 \Rightarrow 9$$

$(0 \oplus 3) \oplus (2 \oplus 4)$ reaches which terms?
Exercise

Recall the following propositions:

\[(0 \oplus 3) \oplus (2 \oplus 4) \Rightarrow 3 \oplus (2 \oplus 4) \Rightarrow 3 \oplus 6 \Rightarrow 9\]

\[(0 \oplus 3) \oplus (2 \oplus 4)\] reaches which terms?

- \[(0 \oplus 3) \oplus (2 \oplus 4) \Rightarrow^* (0 \oplus 3) \oplus (2 \oplus 4)\]
- \[(0 \oplus 3) \oplus (2 \oplus 4) \Rightarrow^* 3 \oplus (2 \oplus 4)\]
- \[(0 \oplus 3) \oplus (2 \oplus 4) \Rightarrow^* 3 \oplus 6\]
- \[(0 \oplus 3) \oplus (2 \oplus 4) \Rightarrow^* 9\]
The normal form of a term

\[ t \text{nfof} t' := t \ \Rightarrow^* \ t' \land \text{nf}(t') \]

What is the normal form of \((0 \oplus 3) \oplus (2 \oplus 4)\)?
The normal form of a term

\[ t \text{ nfof } t' := t \Rightarrow^* t' \land \text{nf}(t') \]

What is the normal form of \((0 \oplus 3) \oplus (2 \oplus 4)\)?

\[(0 \oplus 3) \oplus (2 \oplus 4) \text{ nfof 9}\]

**Definition (normalizing).** We say that a relation, say \(\Rightarrow\), is normalizing if, and only if, we can always find a normal form of \(t\).

Is \(\Rightarrow\) normalizing?
The normal form of a term

\[ t \text{ nfof } t' := t \Rightarrow^* t' \land \text{nf}(t') \]

What is the normal form of \((0 \oplus 3) \oplus (2 \oplus 4)\)?

\((0 \oplus 3) \oplus (2 \oplus 4) \text{ nfof } 9\)

**Definition (normalizing).** We say that a relation, say \(\Rightarrow\), is normalizing if, and only if, we can always find a normal form of \(t\).

Is \(\Rightarrow\) normalizing? **Yes.**

**Theorem.** \(\Rightarrow\) is normalizing.
Normalizing languages

In practice, a normalizing language is one where programs are *guaranteed* to terminate, by design.

Do you know any normalizing language?
Normalizing languages

In practice, a normalizing language is one where programs are \textit{guaranteed} to terminate, by \textit{design}.

Do you know any normalizing language?

\textbf{Yes!} Coq definitions are guaranteed to terminate. The Calculus of Constructions, which Coq implements, enjoys a \textit{normalizing semantics}.

See also \texttt{Dhall}, a configuration programming language, for when you need \textit{controlled} flexibility.
Can we relate small-step semantics and big-step semantics now?
Relating small-step and big-step semantics

- **Theorem 1.** If $t \Downarrow n$, then $t \Rightarrow^* n$. (Homework)

- **Theorem 2.** If $t \mathsf{nf of} t'$, then $\exists n, t' = n \land t \Downarrow n$. (Homework)

*Suggestion:* Regarding Theorem 2, you might want to prove first that if $t \mathsf{nf of} t'$, then $\exists n, t' = n$. And then show that, if $t \Rightarrow^* n$, then $t \Downarrow n$. 
Theorem step_deterministic:
  deterministic step.

Theorem strong_progress : forall t,
  value t \lor (exists t', t \Rightarrow t').

Lemma value_is_nf : forall v,
  value v \rightarrow normal_form step v.

Lemma nf_is_value : forall t,
  normal_form step t \rightarrow value t.

Theorem step_normalizing : (* By induction on [t]. *)
  normalizing step. (* It is crucial to replace a nf by a value. *)
  (* P t1 t2 \Rightarrow* P (C n1) t2 \Rightarrow* P (C n1) (C n2) \Rightarrow* C (n1 + n2) *)
Summary

- Small-step semantics
- Deterministic relations
- Progress
- Normal forms
- Normalizing semantics
- Relating small-step and big-step semantics