CS420

Introduction to the Theory of Computation

Lecture 8: Formal languages

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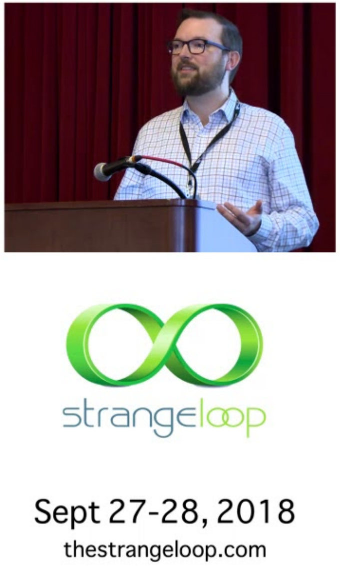
Today we will learn...

- A summary on module 1, intro do module 2
- Formal languages
- A library of languages

A little taste of dependent types



by David Christiansen. URL: `www.youtube.com/watch?v=VxINoKFm-S4`

Note: Σ is `exists`, U is `Prop`, Π is `forall`

What have we learned in Module 1?

1. **A programming language to systematically prove logical facts (Coq)**
   - Dependently-typed language
   - Inductive types
   - Inductive propositions
   - Recursion and the connection to proofs by induction

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2. **Learn from the ground up, by assuming nothing**
   - We defined natural numbers, lists
   - We defined operations on natural numbers, lists (eg, +, -, *)
   - We proved facts about natural numbers, lists (eg, addition is commutative, associative, etc)

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3. **A better understanding of proofs**
   - We can look at a theorem and intuit a proof structure (case analys?, induction?)
   - We can even prove some facts like mindless robots (brute force proofs)

Where are proof assistants used?

- Industry
- Academy
- Education

Where are proof assistants used?

Industry

- CompCert is a C99 compiler written in Coq that is proved correct:
  The ***behavior*** of the output (machine code) is equivalent to that o the source code (C99).
- CompCert is used in avionics and automotive industries

Where are proof assistants used?

Academy

- Programming Language theory
- Parallel Programming theory
- Networks and distributed systems
- Cryptography
- Math (geometry)

What is programming language theory?

Programming Language theory is the cornerstone of computer science

This fields that studies:

- **abstractions of computation** (programming languages, DSLs, APIs, operating systems, distributed systems)
- **PL design & implementation**: compilers, interpreters
- **quality assurance of code** (code analyzers, linters, bug finder)
- **correctness of algorithms** (verification)

Related fields

- Logic
- Software Engineering
- DevOps (automation, DSLs)

Who hires PLT scientists?

Facebook (Automated fault-finding and fixing at Facebook) (ReasonML), Microsoft (Thinking above the code) (C#), Google (Concurrency is not parallelism) (Go, Dart), Amazon (Use of formal methods at AWS), NVidia, Intel, ...

Software Verification Lab

umb-svl.gitlab.io

We model the behavior of intricate systems

- We identify/prove in which cases such intricate systems ***fail*** (eg, data-races being the root causes of deadlocks)
- We build tools that help intricate systems fail less (eg, detecting deadlocks in distributed programs)

Why?

- To tame other people's technology — Marianne Bellotti
- To find bugs without running or even looking at the code — Jay Parlar

Where are proof assistants used?

Education

- To teach programming language theory (Benjamin Pierce, UPenn)
- To teach math (Kevin Buzzard, Imperial College)
- To teach logic
- To teach the theory of computing (here!)

What is next in Module 2?

- Formal languages
- Regular expressions
- Finite State Machines

Formal language

Formal language

> **Insight:** If we restrict what program can do, then what guarantees can we obtain from the restricted program?

- **Goal:** understanding the boundaries of computation
- **Subject:** decision procedures (a form of program)
- **Method:** introducing levels of restrictions in what programs can do

Decision procedures

- **A yes/no question:** that takes a string as input
- **A program:** that implements said question

Formal language examples

Using the mathematical notation, we simply use the **set-builder** notation to represent formal languages. **Set-membership is acceptance:** $x \in L$ reads as L accepts x.

- $L_1 = \{w \mid w \text{ starts with string } 01\}$
  - Examples: $01 \in L_1 \quad 0101 \in L_1 \quad \texttt{foo} \notin L_1$

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- $L_2 = \{w \mid w \text{ contains character } \texttt{a}\}$
  - Examples: $000 \notin L_2 \quad \texttt{aaaaa} \in L_2$

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- $L_3 = \{w \mid w \text{ has } 3 \text{ characters}\}$
  - Examples: $000 \in L_3 \quad \texttt{aa} \notin L_3$
- $L_4 = \{w \mid w \text{ is the textual representation of a prime number }\}$
  - Examples: $\texttt{aa} \notin L_4 \quad 3 \in L_4$

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- $L_4 = \{w \mid w \text{ is the textual representation of a prime number }\}$
  - Examples: $\texttt{aa} \notin L_4 \quad 3 \in L_4$
- $L_5 = \{w \mid w \text{ is a valid C program}\}$
  - Examples: $\texttt{void main}()\{\texttt{return } 0;\} \in L_5 \qquad \texttt{aa} \notin L_5$

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- $L_5 = \{w \mid w \text{ is a valid C program}\}$
  - Examples: $\texttt{void main}()\{\texttt{return } 0;\} \in L_5 \qquad \texttt{aa} \notin L_5$
- $L_6 = \{w \mid w \text{ a valid C program and when run returns code } 0\}$

Looking ahead: formal languages

- Formal languages can be **grouped** and **ordered**
- Smaller languages represent simpler decision problems
- **Insight 1:** we can develop a restricted set of constructs to write all programs in a group
- **Insight 2:** We can know more about simpler languages

Regular ⊂ Context-Free ⊂ Decidable ⊂ Turing Complete

Regular

- $L_1 = \{w \mid w \text{ starts with string } 01\}$
- $L_2 = \{w \mid w \text{ contains character } \texttt{a}\}$
- $L_3 = \{w \mid w \text{ has } 3 \text{ characters}\}$

Context-free

- $L_5 = \{w \mid w \text{ is a valid C program}\}$

Decidable

- $L_4 = \{w \mid w \text{ is a prime number }\}$

Undecidable

- $L_6 = \{w \mid w \text{ a C program and returns code } 0\}$

Formal languages in Coq

How do represent a formal language in Coq?

Formal language

A ***formal language*** is a predicate, of type `(list ascii) → Prop`:

- Takes a **string** (`list ascii`) and returns a **proof object** (an evidence),
- **Acceptance:** We say that the word is **accepted** by language L if, and only if `L w`.

Formal language

A ***formal language*** is a predicate, of type `(list ascii) → Prop`:

- Takes a **string** (`list ascii`) and returns a **proof object** (an evidence),
- **Acceptance:** We say that the word is **accepted** by language L if, and only if `L w`.

Implementation

```
(* Boilerplate code *)
Require Import Coq.Strings.Ascii.
Require Import Coq.Lists.List.
Open Scope char_scope.
Import ListNotations.

(* Definition of a word and a language *)
Definition word := list ascii. (* Think of it as a typedef *)
Definition language := word → Prop.
Definition In w L := L w. (* A word is in the language, if we can show that [L w] holds. *)
```

Strings and their operations

A **string** is a finite sequence of characters. ϵ and $[]$ represent an empty string.

Operators

- **Length:** The length of a string, written $|w|$, is the number of characters that the string contains.
- **Substring:** String z is a substring of w if z appears consecutively within w.
- **Concatenation:** We write $x \cdot y$ for the string concatenation
- **Power:** The power operator x^n where x is a string and n is natural number, defined as x being concatenated n times (yields the empty string when $n = 0$)

$$\texttt{car}^3 = \texttt{carcarcar}$$
$$\texttt{car}^0 = \epsilon$$
$$\texttt{car}^1 = \texttt{car}$$

Strings in Coq

```
Require Import Coq.Strings.Ascii.
Require Import Coq.Lists.List.
Open Scope char_scope.
Import ListNotations.
Require Import Turing.Util.

(* Length: *)
Goal length ["c"; "a"; "r"] = 3. Proof. reflexivity. Qed.
(* Concatenation *)
Goal ["c"] ++ ["a"; "r"] = ["c"; "a"; "r"]. Proof. reflexivity. Qed.
(* Power *)
Goal pow ["c"; "a"; "r"] 3 = ["c"; "a"; "r"; "c"; "a"; "r"; "c"; "a"; "r"].
  Proof. reflexivity. Qed.
Goal pow ["c"; "a"; "r"] 1 = ["c"; "a"; "r"]. Proof. reflexivity. Qed.
Goal pow ["c"; "a"; "r"] 0 = []. Proof. reflexivity. Qed.
```

- Coq has its own string data type, but we are not using that in this course.

Example 1

Recall that `language := word → Prop`

1. Define a language L1 that only accepts word `["c"; "a"; "r"]`
2. Show that L1 accepts `["c"; "a"; "r"]`

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```
Definition L1 w := w = ["c"; "a"; "r"]. (* Define a language L1 *)

(* Show that "car" is in L1 *)
Lemma car_in_l1: In ["c"; "a"; "r"] L1.
Proof.
  unfold L1.
  reflexivity.
Qed.
```

Example 1 (continued)

3. Show that L1 rejects `["f"; "o"; "o"]`

Example 1 (continued)

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```
(* Show that "foo" is not in L1 *)
Lemma foo_not_in_l1: ~ In ["f"; "o"; "o"] L1.
Proof.
```

Example 1 (continued)

3. Show that L1 rejects ["f"; "o"; "o"]

```
(* Show that "foo" is not in L1 *)
Lemma foo_not_in_l1: ~ In ["f"; "o"; "o"] L1.
Proof.

  unfold not, In. (* a proof by contradiction *)
  (* Goal: L1 ["f"; "o"; "o"] -> False *)
  intros N.
  (* N : L1 ["f"; "o"; "o"] *)
  (* Goal: False *)
  unfold L1 in N.
  (* N : ["f"; "o"; "o"] = ["c"; "a"; "r"] *)
  inversion N. (* Explosion principle! *)
Qed.
```

Example 2: `Vowel`

1. Language L2 accepts strings that consist of a single vowel

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```
Definition Vowel w := w = ["a"]
                  \/ w = ["e"]
                  \/ w = ["i"]
                  \/ w = ["o"]
                  \/ w = ["u"].
```

Example 2 (continued)

2. Show that `Vowel` accepts `["a"]`

```
Lemma a_in_vowel: In ["a"] Vowel.
  unfold Vowel.
  Print or.
  (* Inductive or (A B : Prop) : Prop := | or_introl : A → A \/ B *)
  (*                                     | or_intror : B → A \/ B *)
  apply or_introl.
  reflexivity.
Qed.
```

Example 2 (continuation)

3. Show that `Vowel` rejects `["a"; "a"]`

```
Lemma aa_not_in_vowel: ~ In ["a"; "a"] Vowel.
```

Example 2 (continuation)

3. Show that `Vowel` rejects `["a"; "a"]`

```
Lemma aa_not_in_vowel: ~ In ["a"; "a"] Vowel.

  unfold Vowel.
  intros N.
  destruct N as [N|[N|[N|[N|N]]]]; inversion N.
Qed.
```

A library of language operators

A library of language operators

- Recall that our objective is to **group languages**
- We want to have a **compositional** reasoning about languages
- **Idea:** Define an algebra of languages and study how properties behave under this algebra

Language operators

1. `Nil`
2. `Char`
3. `Union`
4. `App`

Nil

> A language that only accepts the empty word.

Set-builder notation: $\{w \mid w = []\}$ or $\{w \mid w = \epsilon\}$

Nil

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```
Definition Nil w := w = [].
```

Correction properties

1. Show that `Nil []`
2. Show that if a word is accepted by `Nil`, then that word must be `[]`

Char

- A language that accepts a single character (given as parameter).

Char

> A language that accepts a single character (given as parameter).

```
Definition Char c (w:word) := w = [c].

Coercion Char: ascii >-> language. (* Allow writing "a" rather than Char "a" *)
```

Correction properties

1. Show that the word `[c]` is accepted by `Char c`: `Char c [c]`
2. Show that any word `w` accepted by `Char c` must be equal to `[c]`

Char

A language that accepts a single character (given as parameter).

```
Definition Char c (w:word) := w = [c].

Coercion Char: ascii >-> language. (* Allow writing "a" rather than Char "a" *)
```

Correction properties

1. Show that the word `[c]` is accepted by `Char c`: `Char c [c]`
2. Show that any word `w` accepted by `Char c` must be equal to `[c]` Show that any word `[c]` is in `Char c`:

Union

- A language that accepts all words of both languages.

Union

A language that accepts all words of both languages.

```
Definition Union (L1 L2:language) w :=
  In w L1 \/ In w L2.

Infix "U" := Union. (* Define a notation for terseness *)
```

Correction properties

1. If the word is accepted by either `L1` or `L2`, then is accepted by `L1 U L2`
2. If the word is accepted by `L1 U L2`, then is accepted by either `L1` or `L2`.

App

- Language L1 >> L2 accepts a word from L1 concatenated with a word from L2

App

Language L1 >> L2 accepts a word from L1 concatenated with a word from L2

```
Definition App (L1 L2:language) w :=
  exists w1 w2, w = w1 ++ w2 /\ L1 w1 /\ L2 w2.
```

Correction properties

1. If w1 in L1 and w2 in L2, then w1 ++ w2 in L1 >> L2.
2. If w in L1 >> L2, then there exists w1 in L1 and w2 in L2 such w = w1 + w2.