# CS420

## Introduction to the Theory of Computation

### Lecture 20: Turing Machines

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# Today we will learn...

- Recap exercises
- TM configuration and configuration history
- TM acceptance
- Variants of Turing Machines
  - Multi-tape
  - Nondeterministic

> Section 3.1, 3.2, and 3.3

## Supplementary material

- Professor Harry Porter's video
- Professor Dan Gusfield's video
- Turing Machines, Stanford Encyclopedia of Philosophy

# Exercises

# Exercise 1

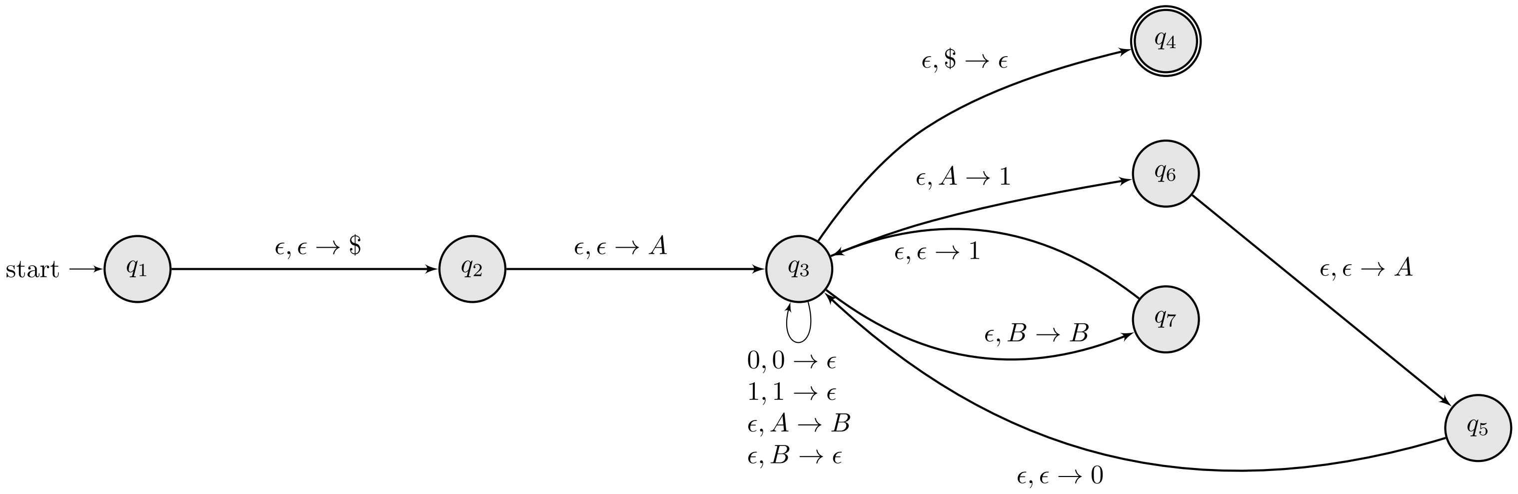
Convert the following grammar into a PDA

$$
\boxed{\begin{aligned} A &\to 0A1 \mid B \\ B &\to 1B \mid \epsilon \end{aligned}}
$$

# Exercise 1

Convert the following grammar into a PDA

$$A \to 0A1 \mid B$$
$$B \to 1B \mid \epsilon$$



# Exercise 2

Show that if $L_1 \cup L_2$ is not context-free, then either $L_1$ is not context-free or $L_2$ is not context free.

**Proof.**

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**Proof.**

1. We know that if $L_1$ is CF and $L_2$ is CF, then $L_1 \cup L_2$ is CF.
2. Apply the contrapositive to (1) and we conclude our proof.

# Exercise 3

We know that $L_2 = \{w \mid w = a^n b^n c^n \vee |w| \text{ is even}\}$ is not context free.

Show that $L_3 = \{a^n b^n c^n \mid n \geq 0\}$ is not context-free without using the Pumping Lemma for CF or the Theorem of non-CF from Lecture 11.

**Proof.**

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**Proof.**

1. It is easy to see that $L_2 = L_3 \cup L_4$ where $L_4 = \{w \mid \; |w| \text{ is even}\}$.

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1. It is easy to see that $L_2 = L_3 \cup L_4$ where $L_4 = \{w \mid \ |w| \text{ is even}\}$.
2. We apply the previous theorem of Exercise 1 and get that either $L_3$ is not context free, or $L_4$ is not-context free.

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2. We apply the previous theorem of Exercise 1 and get that either $L_3$ is not context free, or $L_4$ is not-context free.
3. But we know that $L_4$ is regular and therefore context-free.
4. Thus, $L_2$ is not CF.

# Turing Machine:

configuration & configuration history

# Turing Machines

## Definition 3.3

A Turing machine is a 7-tuple $(Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$

1. $Q$ set of states
2. $\Sigma$ input alphabet not containing the blank symbol $\sqcup$
3. $\Gamma$ the tape alphabet, where $\sqcup \in \Gamma$ and $\Sigma \subseteq \Gamma$
4. $\delta : Q \times \Gamma \to Q \times \Gamma \times \{\mathsf{L}, \mathsf{R}\}$ transition function
5. $q_0 \in Q$ is the start state
6. $q_{accept}$ is the accept state
7. $q_{reject}$ is the reject state ($q_{reject} \neq q_{accept}$)

## To ponder..

- What is the minimum number of states?
- Can the input and the tape alphabets be the same?
- Write a Turing machine with the minimum number of states that recognizes $\emptyset$
- Write a Turing machine with the minimum number of states that recognizes $\Sigma^\star$

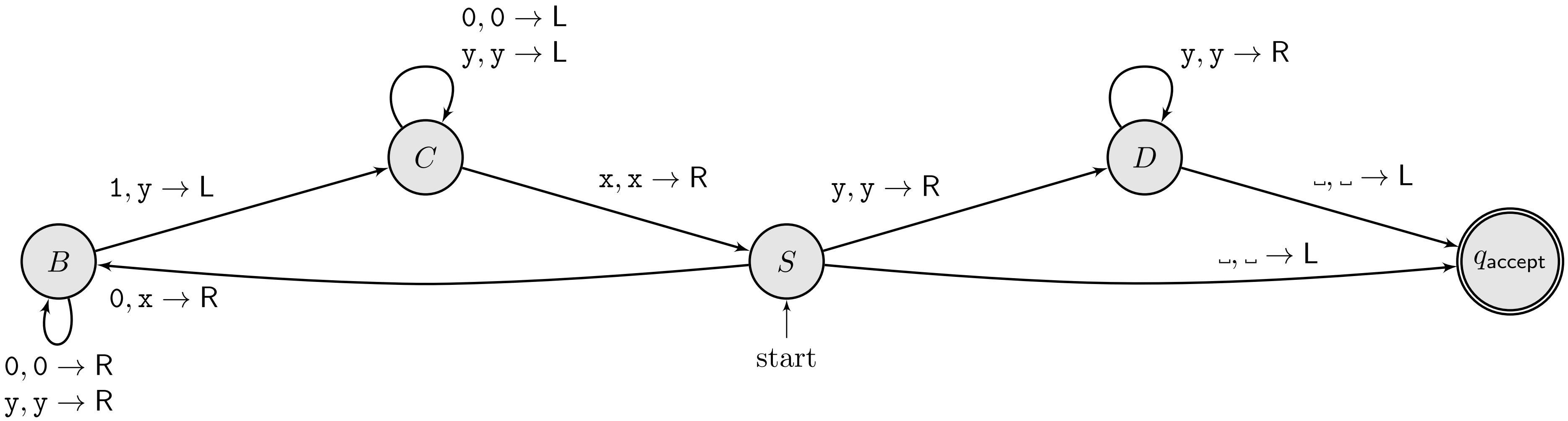
# Configuration

> A configuration is a snapshot of a computation. That is, it contains all information necessary to resume (or replay) a computation from any point in time.

## A configuration consists of

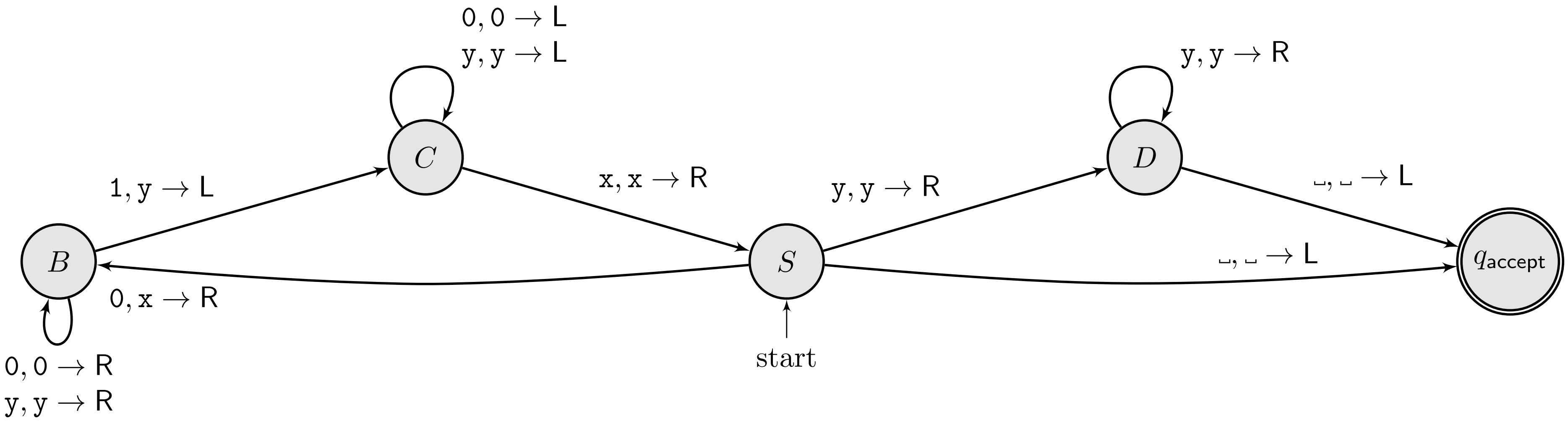
- the tape
- the head of the tape
- the current state

# Example 2




| State | Tape |
|---|---|
| $S$ | 0011 |
| $B$ | X011 |
| $B$ | X011 |
| $C$ | X0Y1 |

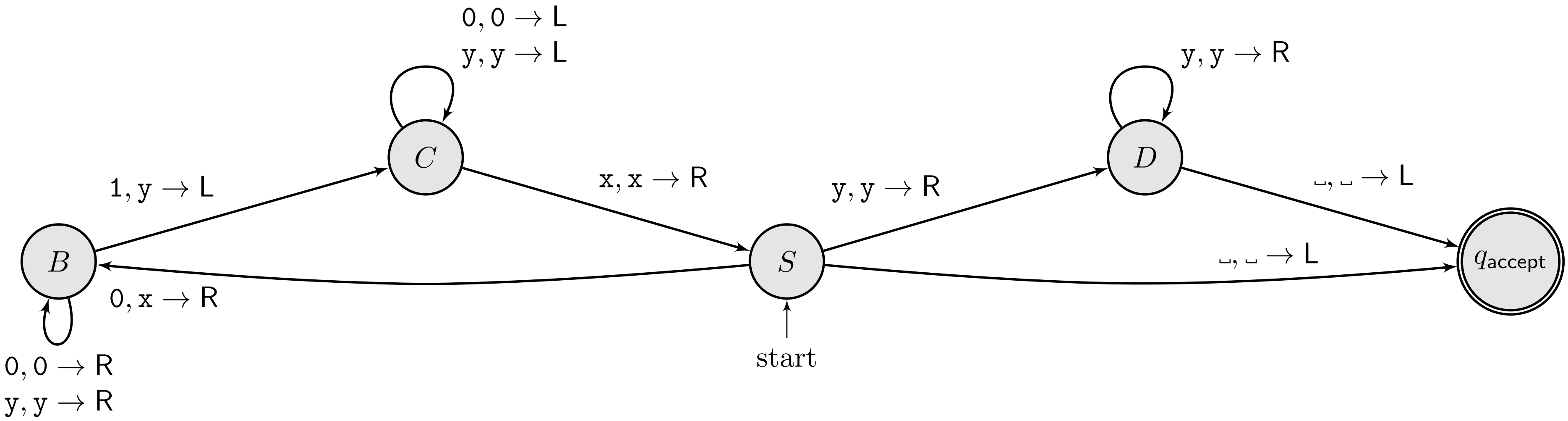
# Example 2




| State | Tape |
|---|---|
| $S$ | 0011 |
| $B$ | X011 |
| $B$ | X011 |
| $C$ | X0Y1 |

| State | Tape |
|---|---|
| $C$ | X0Y1 |
| $S$ | X0Y1 |
| $B$ | XXY1 |
| $B$ | XXY1 |

# Example 2

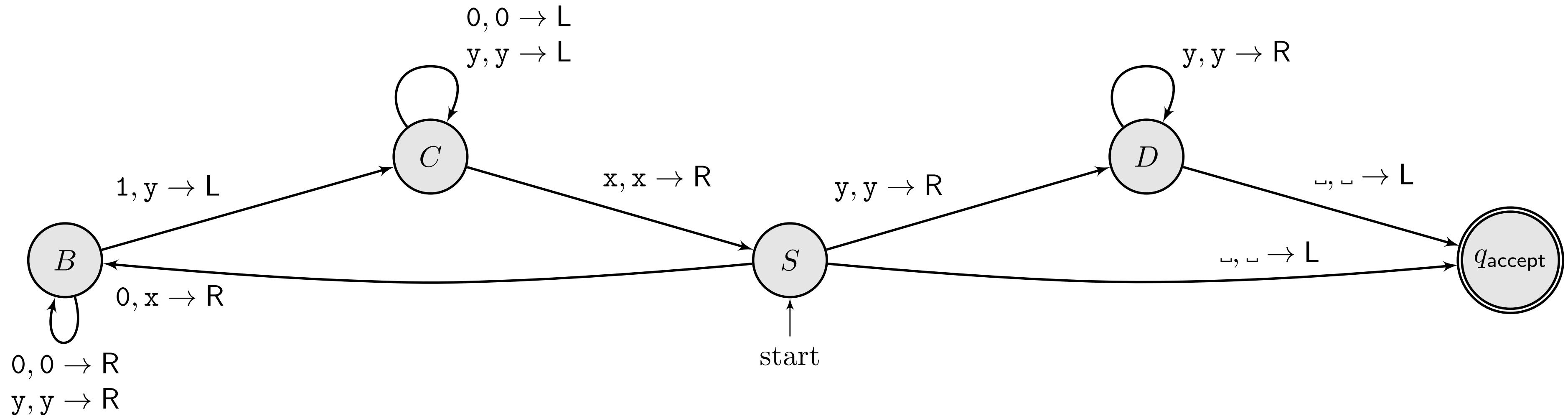



| State | Tape |
|---|---|
| $S$ | 0011 |
| $B$ | X011 |
| $B$ | X011 |
| $C$ | X0Y1 |

| State | Tape |
|---|---|
| $C$ | X0Y1 |
| $S$ | X0Y1 |
| $B$ | XXY1 |
| $B$ | XXY1 |

| State | Tape |
|---|---|
| $C$ | XXYY |
| $C$ | XXYY |
| $S$ | XXYY |
| $D$ | XXYY |

# Example 2




| State | Tape |
|---|---|
| $S$ | 0011 |
| $B$ | X011 |
| $B$ | X011 |
| $C$ | X0Y1 |

| State | Tape |
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| $C$ | X0Y1 |
| $S$ | X0Y1 |
| $B$ | XXY1 |
| $B$ | XXY1 |

| State | Tape |
|---|---|
| $C$ | XXYY |
| $C$ | XXYY |
| $S$ | XXYY |
| $D$ | XXYY |

| State | Tape |
|---|---|
| $D$ | XXYY␣ |

Accept!

Simulate

# Example 1 configuration

| State | Tape | Configuration |
|---|---|---|
| $S$ | 01110 | S 01110 |
| $B$ | x1110 | x B 1110 |
| $B$ | xy110 | xy B 110 |
| $B$ | xyy10 | xyy B 10 |
| $B$ | xyyy0 | xyyy B 0 |
| $B$ | xyyyx_ | xyyyx B |

# Configuration history

> The configuration history (sequence of configurations), describes all configurations from the initial state until a current state.

## Definition

We say that $C_1$ **yields** $C_2$

## Example

| ***Configuration history*** |
|---|
| `S 01110` |
| `x B 1110` |
| `xy B 110` |
| `xyy B 10` |
| `xyyy B 0` |
| `xyyyx B` |

# Acceptance

## A Turing machine

- **accepts** a string if there is a configuration history that reaches the `accept` state.
- **rejects** a string if there is a configuration history that reaches the `reject` state.
- **rejects** a string if it never reaches an `accept` or `reject` states
  This means that for any configuration of any length, there is no accept nor a reject state.

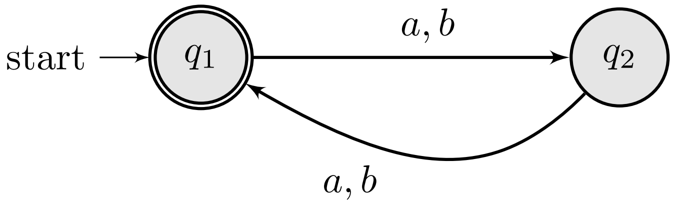
## The acceptance algorithm

- **halts** when the machine is in an `accept` or `reject` state
  ***This is different than NFAs/PDAs which can enter and leave the accept state.***

# Exercise

Give a Turing Machine that recognizes words of an even length.

## NFA



# TM that recognizes words of an even length

(online)

# Exercise

What language does this TM recognize? (online)

# The Church-Turing thesis

# Alan Turing and the Turing Machine

- No computers at the time (1936)
- Alan Turing was researching into the foundations of mathematics
- Original intent: capture all possible processes which can be carried out in computing a number†
- What about non-numerical problems?
- How do Turing machines capture all general and effective procedures which determine whether something is the case or not.

Section 3.3

†: Devise an algorithm that tests whether a polynomial has an integral root.

# The Church-Turing thesis

- Any algorithm can be represented by an equivalent Turing machine
- A problem is computable if, and only if, there exists a Turing Machine that recognizes it.
- Turing Machines are equivalent to $\lambda$-terms

# The Universal Turing Machine

Or, How do we study the limits of computability

# The Universal Turing Machine

A Turing Machine that is capable of simulating any other Turing Machine

- Let $U$ be a TM.
- Given some TM $M$ and some input $w$, we can encoded as an input string, which we represent as $\langle M, w \rangle$

$$U \text{ accepts } \langle M, w \rangle \text{ iff } M \text{ accepts } w$$

Note that the Universal TM is a regular TM. This computability model is expressive enough to simulate itself.

# Alan Turing's impact on modern computers

- Modern computers: von Neumann's EDVAC design
- Fundamental idea of the EDVAC design: stored-programs
  Manipulation of programs as data
- **Universal Turing Machines pioneer the idea of stored programs**

> TM are used to reflect on the limits and potentials of general-purpose computers by engineers, mathematicians and logicians (Module 3)

A single machine simulates all possible machine designs!

Without this idea, computers would have limited scope.

# Multi-tape Turing Machine

# The TM tape only grows to the right

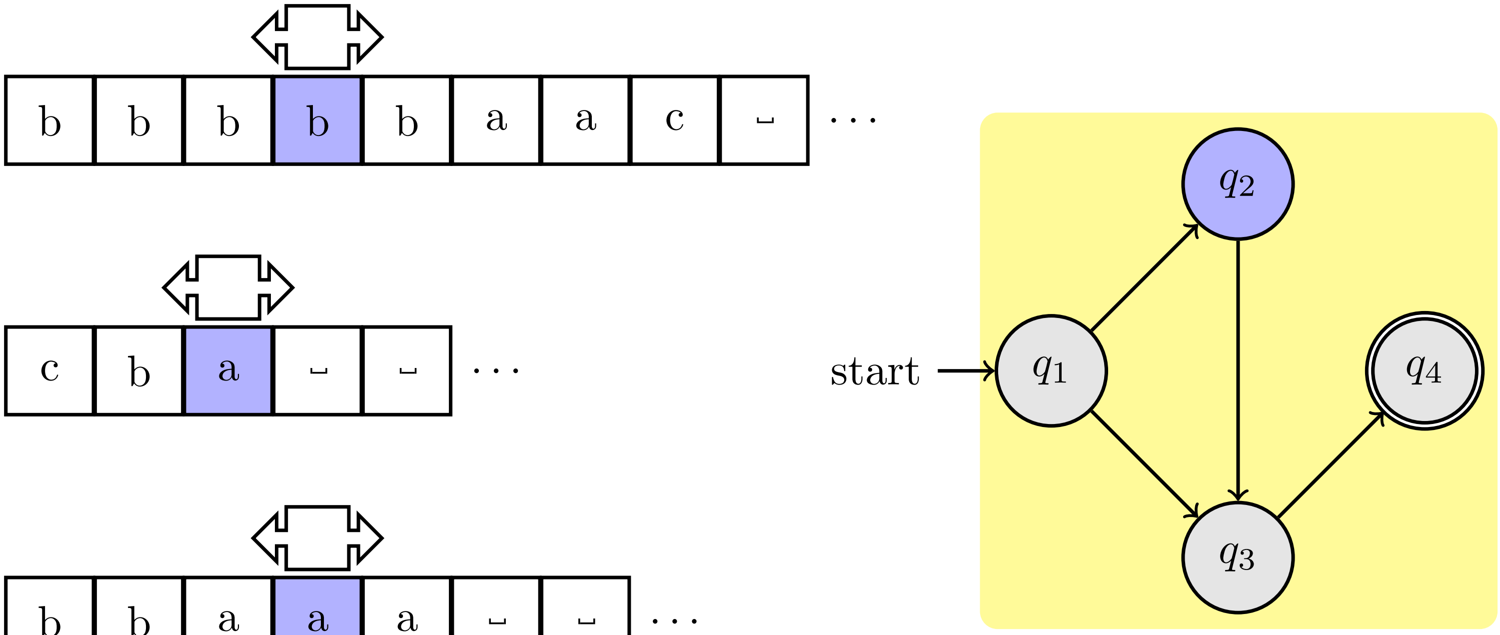
- An important thing to note is that TMs have a tape that grows only to the right
- In `turingmachine.io`, the tape actually grows both ways

Are Turing machines that grow both sides more expressive?

Generalizing, are TM with multi-tapes more expressive?

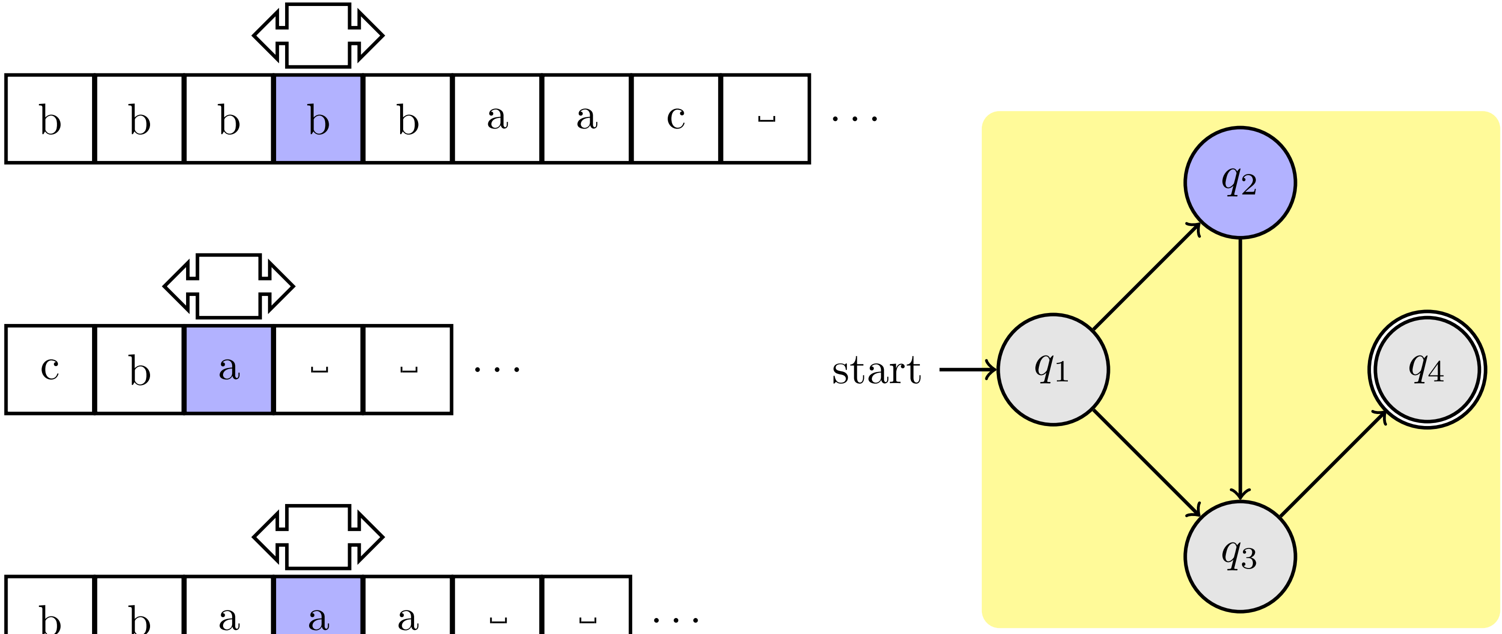
# Multi-tape Turing Machine

- A variation of the Turing Machine with multiple tapes
- The control may issue each head to move: forward, backward, or skip




Multihead Turing Machine

# Are Turing Machines less expressive than Multitape Turing Machines?




Multihead Turing Machine

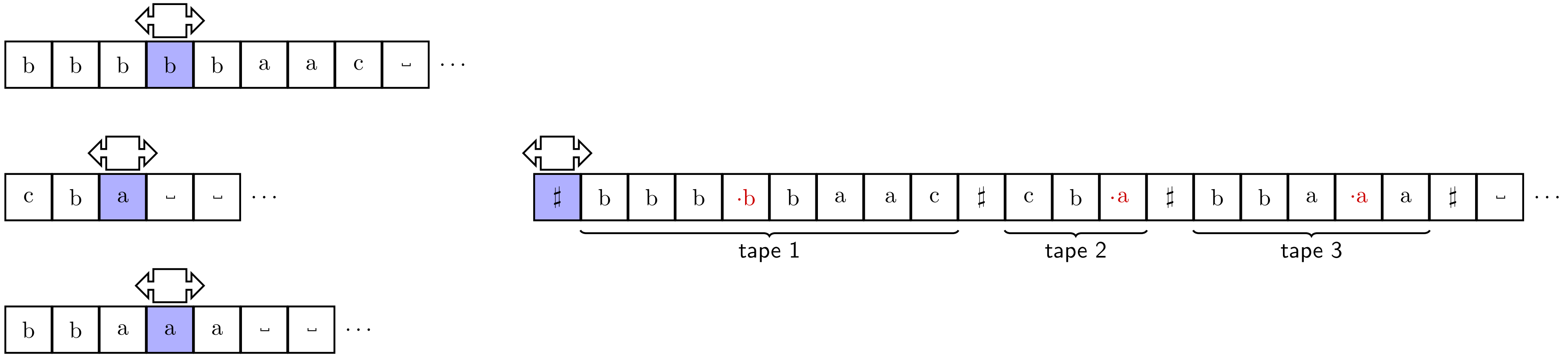
# Turing Machines $\iff$ Multitape Turing Machines

- ($\Leftarrow$) Multitape Turing Machines trivially recognize the same language as Turing Machine (let the number of tapes be 1)
- ($\Rightarrow$) How can a single tape encode multitape?

# Simulating a multitape

## Tape encoding

- Concatenate the three tapes together
- Delimit each tape with a character that is not in the alphabet $\sharp$
- "Tag" the character to encode each tape head (virtual heads), eg $\cdot a$
- The tape head always sits in the beginning of the tape



# Simulating a multitape

## Operation

- To move the $i$-th head, read the tape from the beginning until you read $\sharp$ a total of $i$ times and then seek until you find the marked character
- If the virtual head $i$ hits the end of the tape $\sharp$, then shift the rest of the tape to the right and insert a blank character $\sqcup$

# Nondeterministic Turing Machines

# Nondeterministic Turing Machines (NTM)

A machine can follow more than one transitions for the same input:

$$\delta\colon Q \times \Gamma \to \mathcal{P}(Q \times \Gamma \times \{\mathsf{L}, \mathsf{R}\})$$

## Consequence

> Deterministic can only have one outgoing edge ***per*** character read, a nondeterministic machine can have multiple edges

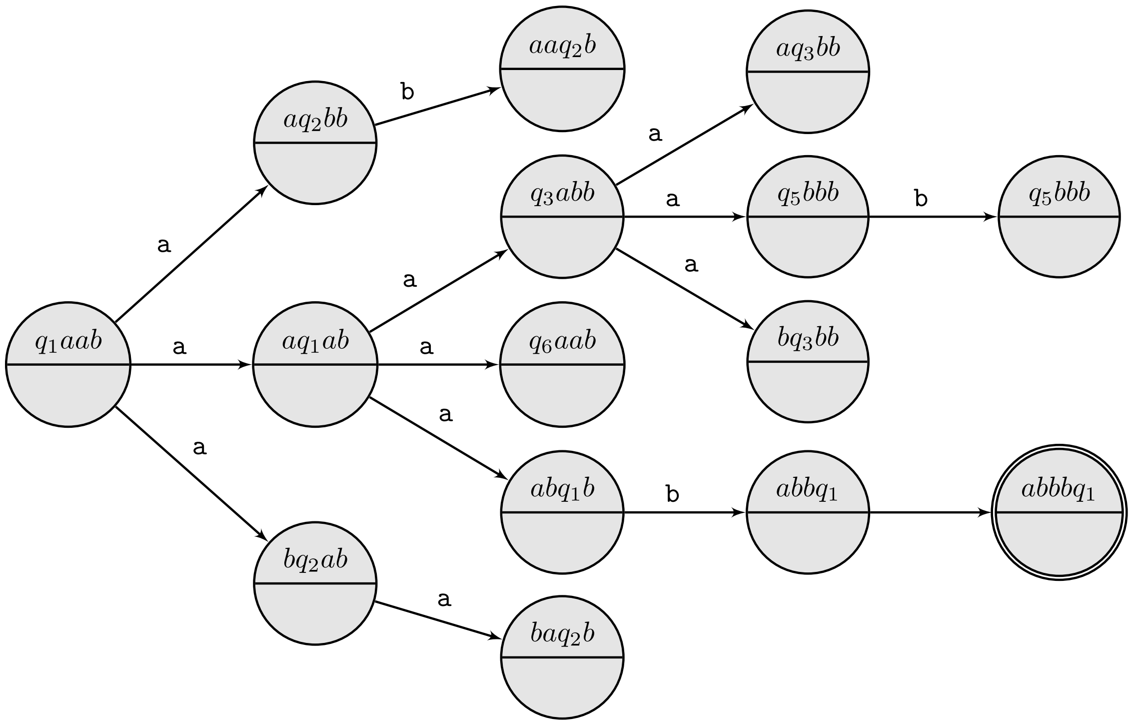
# Configurations in a TM

- In a deterministic TM, a configuration history is ***linear***

```
abc q1 aac → abcx q2 ac → abcxa q2 c → abcx q2 ay
```

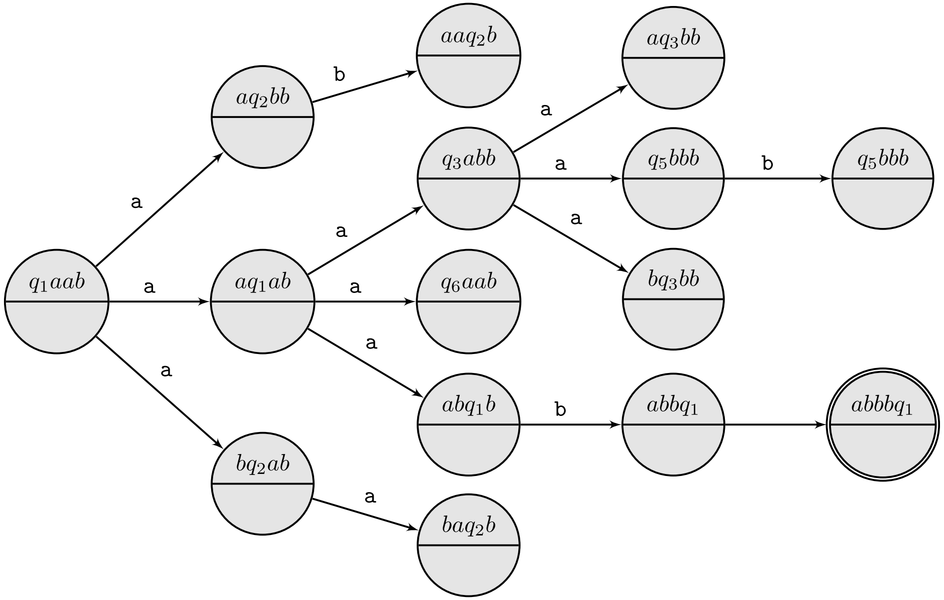
# Configurations in a NTM

- In a nondeterministic TM, a configuration history is a **tree**!



# Nondeterministic Turing Machines

- **Accept:** when **any** branch reaches $q_{accept}$
- **Reject:** when **all** branches reach $q_{reject}$
- To find a single acceptance state we need to search the computation tree



Are Turing Machines less expressive than Nondeterministic Turing Machines?

# TM $\iff$ NTM

- Given an NTM, say $N$ we show how to construct a TM, say $D$
- If $N$ accepts on any branch, then $D$ halts and accepts
- If $N$ rejects on every branch, then $D$ halts and rejects

## Intuition

> Simulate all branches of the computation; search for any node with an accept state.

## Attention!

**Question:** If we are searching a search tree, and there may exist infinite branches (due to loops), how should we search the tree: DFS or BFS?

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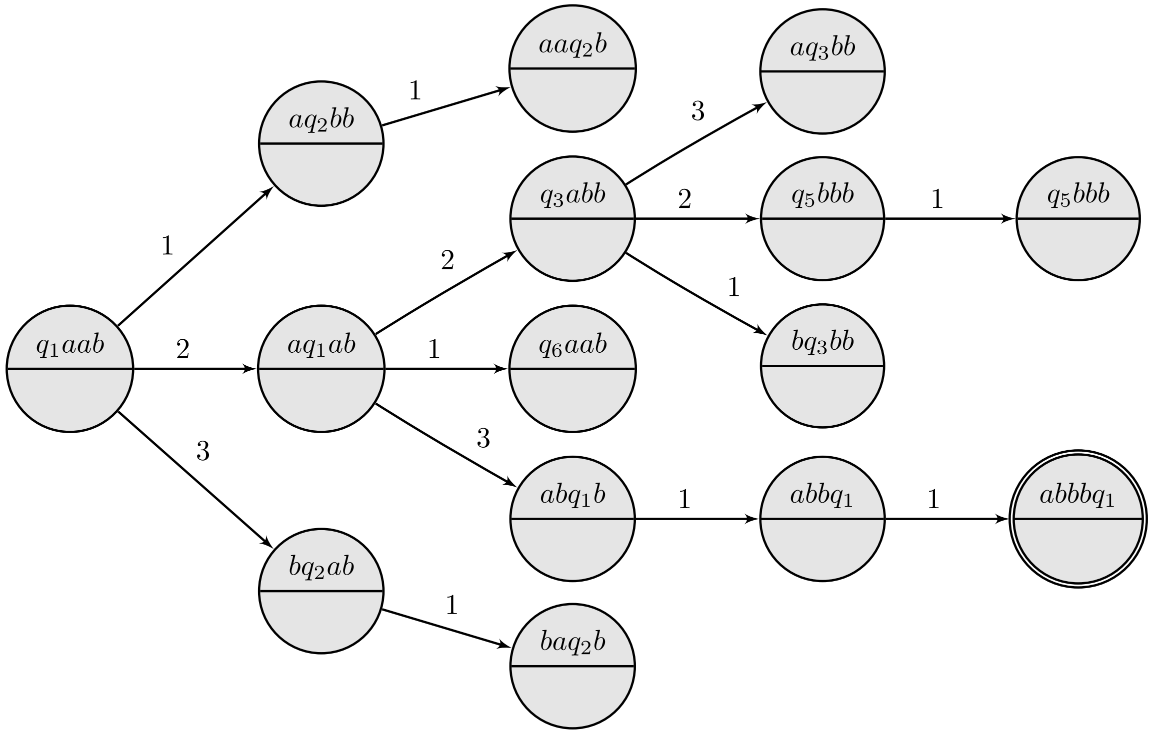
**Bread-First Search** will ensure our search is not caught in a never-ending branch.

# Addressing configuration history

- We can use a sequence of numbers to uniquely identify each node of the configuration history




## Unique paths

- 11
- 223
- 2221
- 221
- 21
- 2311
- 31

# Using a TM to simulate a NTM

## Use 3 tapes

1. **Initial input:** One tape for the input
2. **Simulation tape:** Where we will be executing an address
3. **Address tape:** An ever growing number that uniquely identifies where we are in the tree

## How many choices at each step?

1. Copy tape 1 to tape 2
2. Simulate TM with address from the address tape; if it reaches an accepted state, then ACCEPT, otherwise continue
3. Increment address (next BFS-wise) and go to 1