CS420

Introduction to the Theory of Computation

Lecture 5: Polymorphism; constructor injectivity, explosion principle

Tiago Cogumbreiro

Mini-test 1

- You will have 48 hours to solve it:
  - **Friday 1/Saturday 2?**
  - Saturday 2/Sunday 3?
  - Sunday 3/Monday 4?
  - Monday 4/Tuesday 5?
- I will be give you a sample mini-test as a guide
- You will need to upload a PDF of your solution (either print and write, or use a PDF editor)
- Submission via Gradescope

Today we will learn about...

- Type polymorphism (types in parameters)
- Applying (using) theorems
- Rewriting rules with pre-conditions
- Applying theorems with pre-conditions
- Disjoint constructors
- Principle of explosion

Polymorphism

Recall natlist

```
Inductive natlist : Type :=
  | nil : natlist
  | cons : nat → natlist → natlist.
```

> How do we write a list of `bool`s?

Recall natlist

```
Inductive natlist : Type :=
  | nil : natlist
  | cons : nat → natlist → natlist.
```

How do we write a list of `bools`?

```
Inductive boollist : Type :=
  | bool_nil : boollist
  | bool_cons : bool → boollist → boollist.
```

How to migrate the code that targeted `natlist` to `boollist`? What is missing?

Polymorphism

Inductive types can accept (type) parameters (akin to Java/C# generics, and type variables in C++ templates).

```
Inductive list (X:Type) : Type :=
  | nil : list X
  | cons : X → list X → list X.
```

What is the type of `list`? How do we print `list`?

Constructors of a polymorphic list

```
Check list.
```

yields

```
list
     : Type → Type
```

> What does `Type → Type` mean? What about the following?

```
Search list.
Check list.
Check nil nat.
Check nil 1.
```

How do we encode the list $[1;\ 2]$?

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```
cons nat 1 (cons nat 2 (nil nat))
```

Implement concatenation

```
Fixpoint app (l1 l2 : natlist) : natlist :=
 match l1 with
 | nil => l2
 | h :: t => h :: (app t l2)
 end.
```

How do we make app polymorphic?

Implement concatenation

```
Fixpoint app (l1 l2 : natlist) : natlist :=
 match l1 with
 | nil => l2
 | h :: t => h :: (app t l2)
 end.
```

> How do we make app polymorphic?

```
Fixpoint app (X:Type) (l1 l2 : list X) : list X :=
 match l1 with
 | nil _ => l2
 | cons _ h  t => cons X h (app X t l2)
 end.
```

> What is the type of app?

Implement concatenation

```
Fixpoint app (l1 l2 : natlist) : natlist :=
 match l1 with
 | nil => l2
 | h :: t => h :: (app t l2)
 end.
```

How do we make `app` polymorphic?

```
Fixpoint app (X:Type) (l1 l2 : list X) : list X :=
 match l1 with
 | nil _ => l2
 | cons _ h  t => cons X h (app X t l2)
 end.
```

What is the type of `app`? `forall X : Type, list X → list X → list X`

Type inference (1/2)

Coq infer type information:

```
Fixpoint app X l1 l2 :=
 match l1 with
 | nil _ => l2
 | cons _ h  t => cons X h (app X t l2)
 end.

Check app.
```

outputs

```
app
     : forall X : Type, list X -> list X -> list X
```

Type inference (2/2)

```
Fixpoint app X (l1 l2:list X) :=
 match l1 with
 | nil _ => l2
 | cons _ h  t => cons _ h (app _ t l2)
 end.

Check app.

app
     : forall X : Type, list X -> list X -> list X
```

Let us look at the output of

```
Compute cons nat 1 (cons nat 2 (nil nat)).
Compute cons _ 1 (cons _ 2 (nil _)).
```

Type information redundancy

- If Coq can infer the type, can we automate inference of type parameters?

Type information redundancy

- If Coq can infer the type, can we automate inference of type parameters?

```
Fixpoint app {X:Type} (l1 l2:list X) : list X :=
 match l1 with
 | nil ⇒ l2
 | cons h  t ⇒ cons h (app t l2)
 end.
```

Alternatively, use `Arguments` after a definition:

```
Arguments nil {X}.       (* braces should surround argument being inferred *)
Arguments cons {_} _ _.  (* you may omit the names of the arguments *)
Arguments app {X} l1 l2. (* if the argument has a name, you *must* use the *same* name *)
```

Try the following

```
Inductive list (X:Type) : Type :=
  | nil : list X
  | cons : X -> list X -> list X.

Arguments nil {_}.
Arguments cons {X} x y.

Search list.
Check list.
Check nil nat.
Compute nil nat.
```

> What went wrong?

Try the following

```
Inductive list (X:Type) : Type :=
 | nil : list X
 | cons : X → list X → list X.

Arguments nil {_}.
Arguments cons {X} x y.

Search list.
Check list.
Check nil nat.
Compute nil nat.
```

> What went wrong? How do we supply type parameters when they are being automatically inferred?

Try the following

```
Inductive list (X:Type) : Type :=
  | nil : list X
  | cons : X → list X → list X.

Arguments nil {_}.
Arguments cons {X} x y.

Search list.
Check list.
Check nil nat.
Compute nil nat.
```

> What went wrong? How do we supply type parameters when they are being automatically inferred?

Prefix a definition with `@`. Example: `@nil nat.`

Tactics.v

Exercise 1: transitivity over equals

```
Theorem eq_trans : forall (T:Type) (x y z : T),
  x = y → y = z → x = z.
Proof.
  intros T x y z eq1 eq2.
  rewrite → eq1.
```

yields

```
1 subgoal
T : Type
x, y, z : T
eq1 : x = y
eq2 : y = z
______________________________________(1/1)
y = z
```

How do we conclude this proof?

Exercise 1: transitivity over equals

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Theorem eq_trans : forall (T:Type) (x y z : T),
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```
1 subgoal
T : Type
x, y, z : T
eq1 : x = y
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______________________________________(1/1)
y = z
```

> How do we conclude this proof? Yes, `rewrite -> eq2. reflexivity.` works.

Exercise 1: introducing `apply`

Apply takes an hypothesis/lemma to conclude the goal.

```
  apply eq2.
Qed.
```

`apply` takes `?X` to conclude a goal `?X` (resolves `forall`s in the hypothesis).

```
1 subgoal
T : Type
x, y, z : T
eq1 : x = y
eq2 : y = z
______________________________________(1/1)
y = z
```

Applying conditional hypothesis

`apply` uses an hypothesis/theorem of format `H1 → ... → Hn → G`, then solves goal `G`, and produces new goals `H1`, ..., `Hn`.

```
Theorem eq_trans_2 : forall (T:Type) (x y z: T),
  (x = y → y = z → x = z) → (* eq1 *)
  x = y →                   (* eq2 *)
  y = z →                   (* eq3 *)
  x = z.
Proof.
  intros T x y z eq1 eq2 eq3.
  apply eq1.   (* x = y → y = z → x = z *)
```

(Done in class.)

Rewriting conditional hypothesis

`apply` uses an hypothesis/theorem of format `H1 → ... → Hn → G`, then solves goal `G`, and produces new goals `H1`, ..., `Hn`.

```
Theorem eq_trans_3 : forall (T:Type) (x y z: T),
  (x = y -> y = z -> x = z) ->  (* eq1 *)
  x = y ->                       (* eq2 *)
  y = z ->                       (* eq3 *)
  x = z.
Proof.
  intros T x y z eq1 eq2 eq3.
  rewrite -> eq1.  (* x = y -> y = z -> x = z *)
```

(Done in class.)

> Notice that there are 2 conditions in `eq1`, so we get 3 goals to solve.

Recap

What's the difference between `reflexivity`, `rewrite`, and `apply`?

1. `reflexivity` solves ***goals*** that can be simplified as an equality like `?X = ?X`
2. `rewrite -> H` takes an ***hypothesis*** `H` of type `H1 -> ... -> Hn -> ?X = ?Y`, finds any sub-term of the goal that matches `?X` and replaces it by `?Y`; it also produces goals `H1`,..., `Hn`. `rewrite` does not care about what your goal is, just that the goal **must** contain a pattern `?X`.
3. `apply H` takes an hypothesis `H` of type `H1 -> ... -> Hn -> G` and solves ***goal*** `G`; it creates goals `H1`, ..., `Hn`.

Apply with/Rewrite with

```
Theorem eq_trans_nat : forall (x y z: nat),
  x = 1 ->
  x = y ->
  y = z ->
  z = 1.
Proof.
  intros x y z eq1 eq2 eq3.
  assert (eq4: x = z). {
    apply eq_trans.
```

outputs

```
Unable to find an instance for the variable y.
```

We can supply the missing arguments using the keyword `with`: `apply eq_trans with (y:=y).`

> Can we solve the same theorem but use `rewrite` instead?

Symmetry

What about this exercise?

```
Theorem eq_trans_nat : forall (x y z: nat),
  x = 1 →
  x = y →
  y = z →
  1 = z.
Proof.
  intros x y z eq1 eq2 eq3.
  assert (eq4: x = z). {
```

Symmetry

What about this exercise?

```
Theorem eq_trans_nat : forall (x y z: nat),
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  x = y →
  y = z →
  1 = z.
Proof.
  intros x y z eq1 eq2 eq3.
  assert (eq4: x = z). {
```

We can rewrite a goal `?X = ?Y` into `?Y = ?X` with `symmetry`.

Apply in example

```
Theorem silly3' : forall (n : nat),
  (beq_nat n 5 = true → beq_nat (S (S n)) 7 = true) →
  true = beq_nat n 5  →
  true = beq_nat (S (S n)) 7.
Proof.
  intros n eq H.
  symmetry in H.
  apply eq in H.
```

(Done in class.)

Targetting hypothesis

- `rewrite -> H1 in H2`
- `symmetry in H`
- `apply H1 in H2`

Forward vs backward reasoning

If we have a theorem `L: C1 → C2 → G`:

- ***Goal takes last:*** apply to goal of type `G` and replaces `G` by `C1` and `C2`
- ***Assumption takes first:*** apply to hypothesis `L` to an hypothesis `H: C1` and rewrites `H:C2 → G`

Proof styles:

- ***Forward reasoning***: (`apply in` hypothesis) manipulate the hypothesis until we reach a goal.
  Standard in math textbooks.
- ***Backward reasoning***: (`apply` to goal) manipulate the goal until you reach a state where you can apply the hypothesis.
  Idiomatic in Coq.

Recall our encoding of natural numbers

```
Inductive nat : Type :=
  | O : nat
  | S : nat → nat.
```

1. Does the equation `S n = 0` hold? Why?

Recall our encoding of natural numbers

```
Inductive nat : Type :=
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1. Does the equation `S n = 0` hold? Why?
   No the constructors are implicitly disjoint.
2. If `S n = S m`, can we conclude something about the relation between `n` and `m`?

Recall our encoding of natural numbers

```
Inductive nat : Type :=
  | O : nat
  | S : nat → nat.
```

1. Does the equation `S n = 0` hold? Why?
   No the constructors are implicitly disjoint.
2. If `S n = S m`, can we conclude something about the relation between `n` and `m`?
   Yes, constructor `S` is injective. That is, if `S n = S m`, then `n = m` holds.

> These two principles are available to all inductive definitions! How do we use these two properties in a proof?

Proving that S is injective (1/2)

```
Theorem S_injective : forall (n m : nat),
  S n = S m →
  n = m.
Proof.
  intros n m eq1.
  inversion eq1.
```

If we run `inversion`, we get:

```
1 subgoal
n, m : nat
eq1 : S n = S m
H0 : n = m
______________________________________(1/1)
m = m
```

Injectivity in constructors

```
Theorem S_injective : forall (n m : nat),
  S n = S m →
  n = m.
Proof.
  intros n m eq1.
  inversion eq1 as [eq2].
```

If you want to name the generated hypothesis you must figure out the destruction pattern and use as `[...]`. For instance, if we run `inversion eq1 as [eq2]`, we get:

```
1 subgoal
n, m : nat
eq1 : S n = S m
eq2 : n = m
______________________________________(1/1)
m = m
```

Disjoint constructors

```
Theorem beq_nat_0_l : forall n,
   beq_nat 0 n = true -> n = 0.
Proof.
  intros n eq1.
  destruct n.
```

(To do in class.)

Principle of explosion

Ex falso (sequitur) quodlibet

`inversion` concludes absurd hypothesis, where there is an equality between different constructors. Use `inversion eq1` to conclude the proof below.

```
1 subgoal
n : nat
eq1 : false = true
______________________________________(1/1)
S n = 0
```