CS420

Introduction to the Theory of Computation

Lecture : Module 2 recap

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Disclaimer: PDA semantics

- The PDAs we learned are slightly different than the ones in the book
- The definition of PDAs we introduced in our lecture have two special operations (stack-empty? and clear-stack); in the book the semantics are a bit simpler, but the diagrams become a bit more verbose
- This simplifies the design of PDAs
- Our version is a super-set of the book (in terms of state-diagram), so the examples in the book should all have the same meaning
- You can use either version in Mini-Test 2

Mini-Test 2

- Written exam: 80 points (out of 100)
- Coq script: 35 points (out of 100)
- Total: 115 points (15 extra points max)

About

- 13 questions
- Max value per question: 15 points
- Took me 20 minutes to solve

Exercise 1

$$A \to 0A1 \mid B$$
$$B \to 1A \mid \epsilon$$

Exercise 3 of Lesson 10

Convert the following grammar into a PDA

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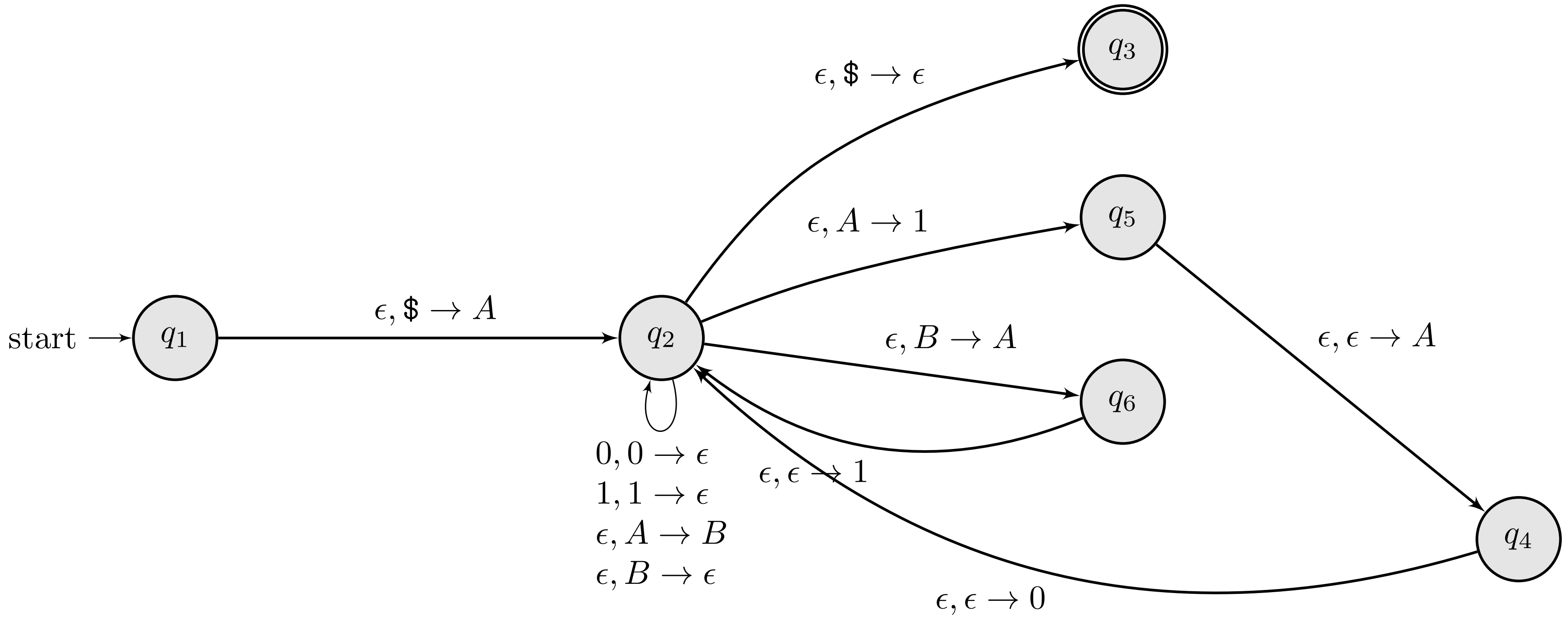

Exercise 2

Given that $L_2 = \{0^i 1^j 0^k \mid i = j = k \vee k = i + j\}$ is not context-free, show that $\{0^n 1^n 0^n \mid n \geq 0\}$ is not context-free without using the pumping lemma.

Exercise 2

Given that $L_2 = \{0^i 1^j 0^k \mid i = j = k \vee k = i + j\}$ is not context-free, show that $\{0^n 1^n 0^n \mid n \geq 0\}$ is not context-free without using the pumping lemma.

1. $L_2 = A \cup B$ where $A = \{0^i 1^j 0^k \mid k = i + j\}$ and $B = \{0^n 1^n 0^n \mid n \geq 0\}$
2. We know that if L CF and L' CF, then $L \cup L'$ CF.
3. Applying the contra-positive we have that: $L \cup L'$ not CF implies that not (L is CF and L' CF)
4. Since $\neg(P \wedge Q) \implies \neg P \vee \neg Q$, thus either A not CF or B not CF.
5. We have that B is CF (from HW5), hence A not CF

> These exercises always use contra-positive of the closure properties (union, star, concat)

You are asked to show that given L is not context-free, then L' is not context free:

1. Show that L' can be obtained from L using union/concat/star
2. Apply the contra-positive of the closure property
3. Conclude the goal

Exercise 3

Let be L context-free. Is $L_3 = L^\star$

- regular?
- not-regular?
- context-free?
- not-context-free?

If L_3 is context-free, prove it.

If L_3 is not-context-free, prove it.

Exercise 3

Let be L context-free. Is $L_3 = L^\star$

- regular?
- not-regular?
- context-free?
- not-context-free?

If L_3 is context-free, prove it. **Solution:** L_3 is context-free, because $L_3 = L^\star$ and we know that if L is context-free, then L_3 is context-free.

If L_3 is not-context-free, prove it.

Exercise 4

For any language L_1 there exists a regular language L_2 such that $L_1 \subseteq L_2$.

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For any language L_1 there exists a regular language L_2 such that $L_1 \subseteq L_2$.

True.

Proof. Let $L_2 = \Sigma^\star$ we have that $\forall L, L \subseteq \Sigma^\star$, thus $L_1 \subseteq L_2$.

Key takeaways

Any language (e.g., not-regular, or not-context free) is "between" two regular languages

$$\underbrace{\emptyset}_{\text{regular}} \subseteq L \subseteq \underbrace{\Sigma^\star}_{\text{regular}}$$

- There exists a regular language such that no other language is strictly smaller
- Conversely, there exists a language such that no other language is strictly larger

Exercise 5

A Turing machine's head can stay in the same position in two consecutive steps.

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A Turing machine's head can stay in the same position in two consecutive steps.

Solution

True. If the head of the tape is on the beginning of the tape (left-most) and moves left, the head remains in the same position in the following step.

Exercise 6

- A, B regular
- C, D not-regular + context-free
- E, F not-regular + not-context-free

Check all that apply, or use a question-mark if it is not possible to know, in which case give a positive and a negative example.

	A	C	E	$A \cup B$	$A \cup C$	$C \cup D$	$D \cup E$
Regular							
Not-regular							
Context-free							
Not-context free							
Turing-Recognizable							

Exercise 7

- A, B regular
- C, D not-regular + context-free
- E, F not-regular + not-context-free

	A	C	E	$A \cup B$	$A \cup C$	$C \cup D$	$D \cup E$
Regular	✓			✓	?	?	?
Not-regular		✓			?	?	?
Context-free	✓	✓	✓	✓	✓	✓	?
Not-context free			✓				?

- $A \cup B$: $\emptyset \cup \{0^n 1^n \mid n \geq 0\}$ not regular; $\Sigma^\star \cup \{0^n 1^n \mid n \geq 0\}$ regular
- $C \cup D$: $\{0^i 1^j \mid i < j\} \cup \{0^i 1^j \mid i \geq j\} = \mathcal{L}(0^\star 1^\star)$ regular;
  $\{0^i 1^j \mid i < j\} \cup \{0^n 1^n \mid n \geq 0\} = \{0^i 1^j \mid i \leq j\}$
- Same idea for $D \cup E$

Tip

> $\cup$ on REG/CTX is akin to $+$ on INT/LONG

- INT + LONG = LONG, similarly, REG $\cup$ CTX = CTX
- INT + INT = INT, similarly REG $\cup$ REG = REG
- Any INT is a LONG, similarly any REG is CTX

The analogy breaks with not-reg/not-ctx

Exercise 8

Spot the error

	A	B	C	D
Regular	✓	✓	✓	
Not-regular	✓			✓
Context-free	✓	✓		✓
Not-context free			✓	✓
Turing-Recognizable	✓		✓	✓

Exercise 8

Spot the error

	A	B	C	D
Regular	✓	✓	✓	
Not-regular	✓			✓
Context-free	✓	✓		✓
Not-context free			✓	✓
Turing-Recognizable	✓		✓	✓

Solution

Language	***Justification***
A	A language cannot be regular and not regular
B	Regular/CF language is Turing recognizable
C	A regular language has to be CF
D	A language cannot be CF and not-CF

- We have not learned about non-Turing recognizable languages (Module 3), so they won't show up in Module 2

Exercise 9

> Prove or disprove the following statement.

If L is not regular, then L is not context-free.

Exercise 9

Prove or disprove the following statement.

If L is not regular, then L is not context-free.

Answer: False. $\{0^n 1^n \mid n \geq 0\}$ is not-regular, yet context-free (lesson 7).

What does disproving a statment work?

Disproving P means showing that $\neg P$ holds, i.e., having P leads to contradiction.

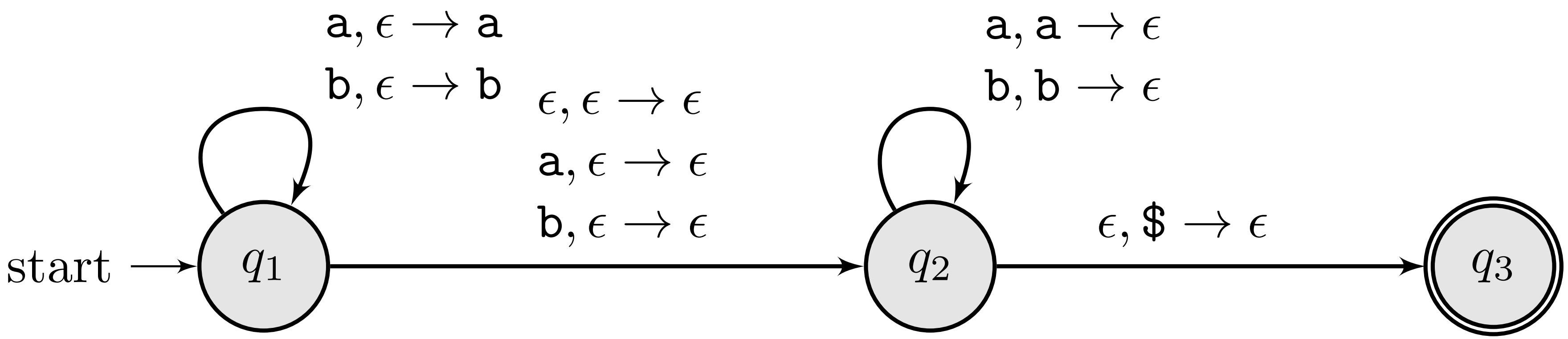
```
Variable _0n1n : lang.
Axiom _0n1n_not_reg: ~ Reg _0n1n.
Axiom _0n1n_cf: CtxFree _0n1n.
Lemma not_reg_to_not_cf:
  ~ (forall l, ~ Reg l -> ~ CtxFree l).
Proof.
  (* Proof follows by contradiction *)
  intros N.                (* Assume N: forall l, ~ Reg l -> CtxFree l *)
  assert (N := N _0n1n).   (* Let us instantiate l to be 0^n 1^n in N (contra-example) *)
  assert (H := _0n1n_not_reg). (* We know that H: 0^n 1^n not regular *)
  apply N in H.            (* Hence, 0^n 1^n not context-free *)
  contradict H.            (* Which we contradict... *)
  apply _0n1n_cf.          (* ... because 0^n 1^n i context-free. *)
Qed.
```

What does disproving a statement work?

Essentially, disproving P actually means showing that $\neg\forall x, P(x)$ holds and the goal is giving a counter-example x such that $P(x)$ leads to a contradiction.

- $\neg\forall x, P(x) \iff \exists x, \neg P(x)$
- $\exists x, \neg P(x)$ and since we know $P(x)$, then we reach a contradiction

Exercise 10: rejects abb



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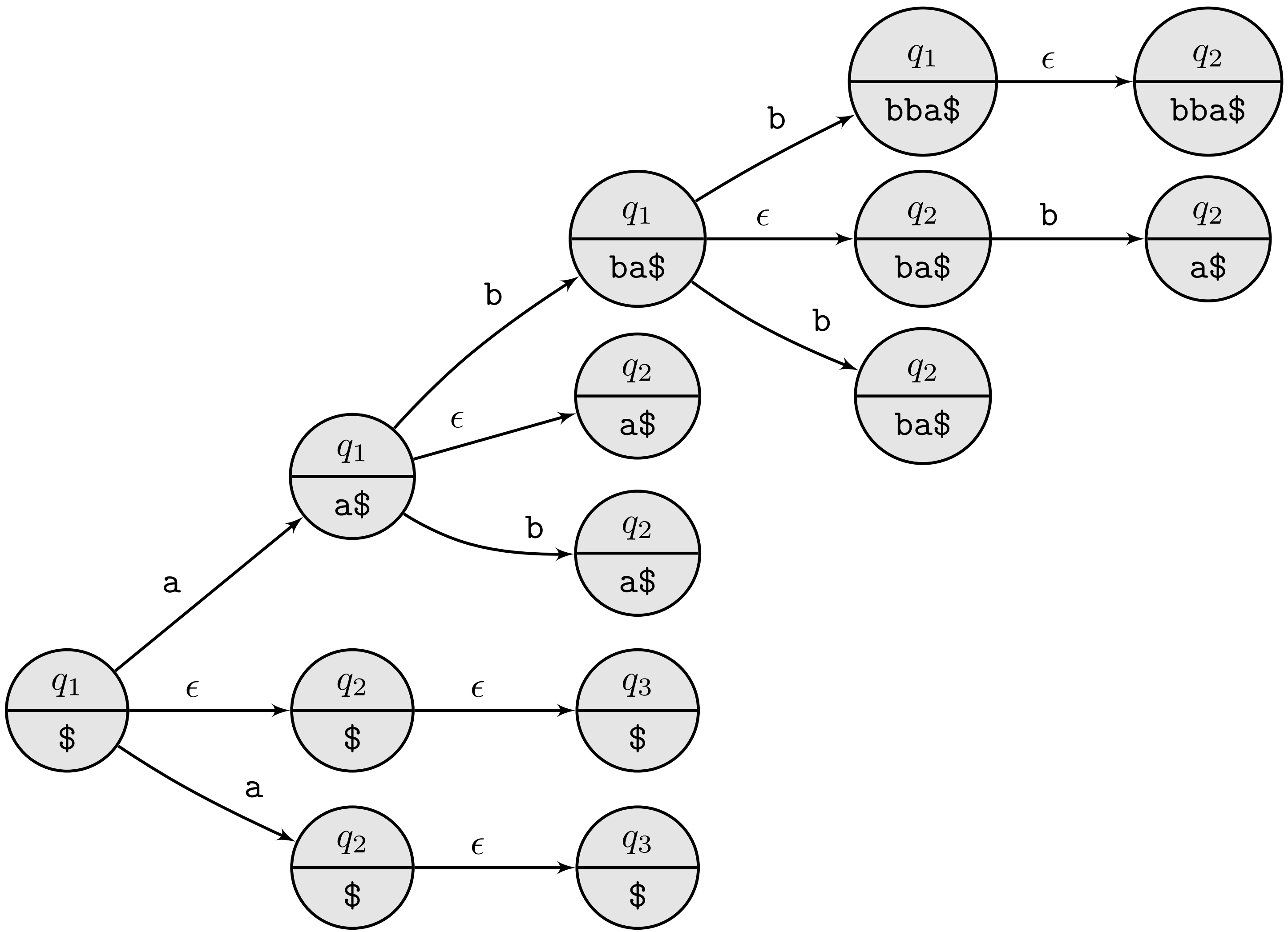
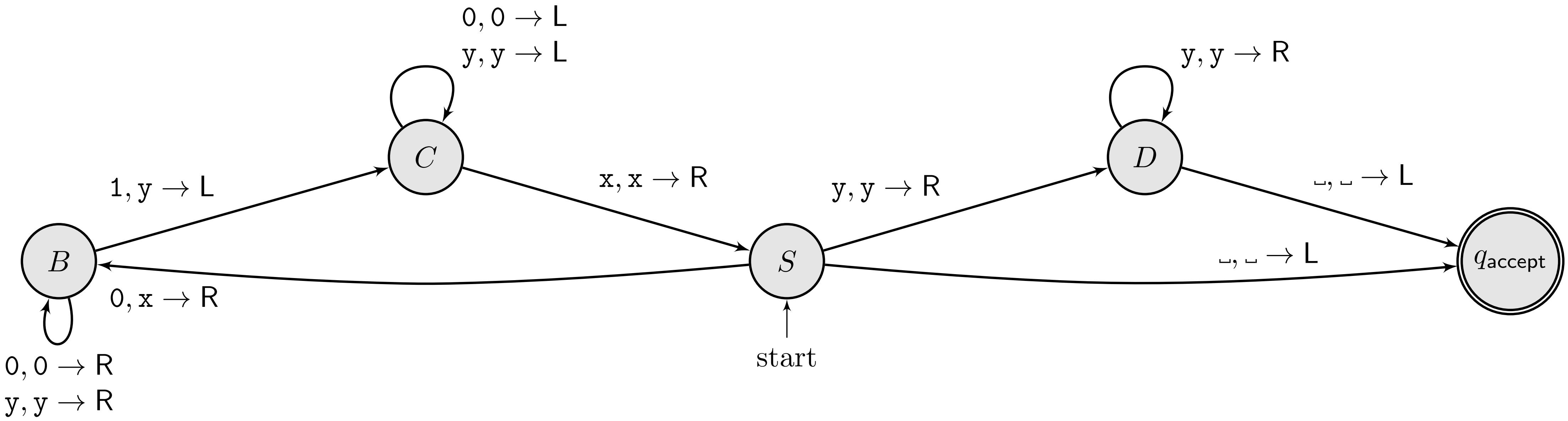


Example 11



- Give the configuration history for the smallest string the TM accepts
- Given configuration $XC0Y1$, what is the next configuration

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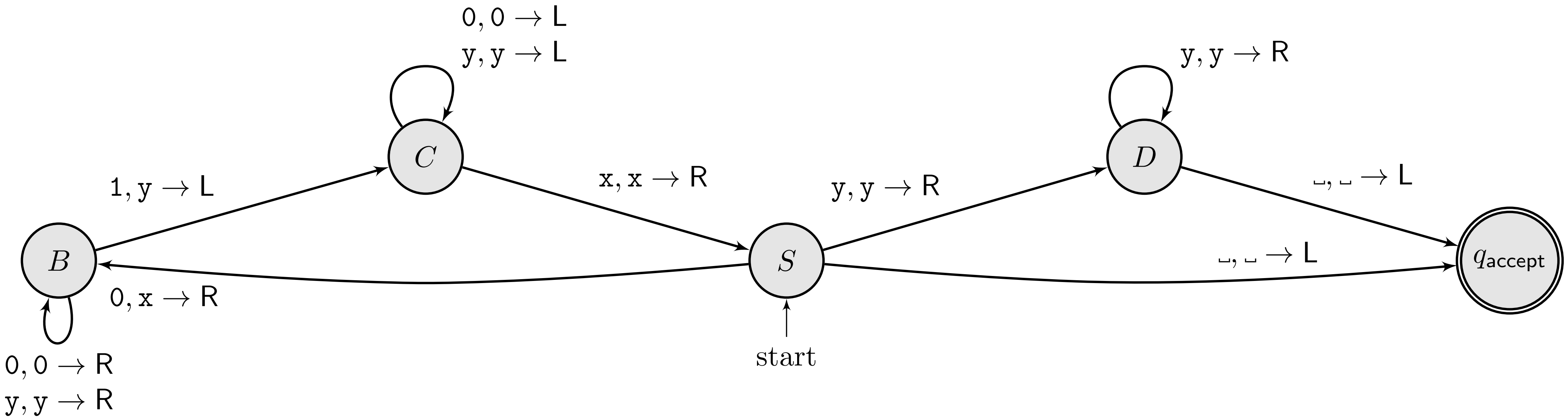

- Give the configuration history for the smallest string the TM accepts:
  $S\square$
  $q_{accept}\square$
- Given configuration $XC0Y1$, what is the next configuration: $CX0Y1$