CS420

Introduction to the Theory of Computation

Lecture 21: Undecidability and unrecognizability

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Today we learn

- Decidability results
- Halting problem
- Emptiness for TM is undecidable

Section 4.2, 5.1

Decidability and Recognizability

Understanding the limits of decision problems

> ***Implementation:*** algorithm that answers a decision problem, that is algorithm says YES whenever decision problem says YES.

- **Decidability:** there is an implementation that terminates for all inputs
- **Undecidability:** any implementation will loop for some inputs
- **Unrecognizability:** no implementation is possible

Decidability and Recognizability

Understanding the limits of decision problems

> ***Implementation:*** algorithm that answers a decision problem, that is algorithm says YES whenever decision problem says YES.

- **Decidability:** there is an implementation that terminates for all inputs
- **Undecidability:** any implementation will loop for some inputs
- **Unrecognizability:** no implementation is possible

Technically we are learning

- Proving the correctness of algorithms
- Proving the termination of algorithms
- Proving non-trivial results (combining multiple theorems)

Corollary 4.23

$\overline{A_{TM}}$ is unrecognizable

Corollary 4.23: $\overline{A}_{TM}$ is unrecognizable

```
Lemma co_a_tm_not_recognizable:
  ~ Recognizable (compl A_tm).
```

Done in class...

Corollary 4.18

Some languages are unrecognizable

Corollary 4.18 Some languages are unrecognizable

Proof.

Corollary 4.18 Some languages are unrecognizable

Proof. An example of an unrecognizable language is: $\overline{A_{TM}}$

If L is decidable,

then $\overline{L}$ is decidable

On pen-and-paper proofs

THEOREM 4.22

A language is decidable iff it is Turing-recognizable and co-Turing-recognizable.

In other words, a language is decidable exactly when both it and its complement are Turing-recognizable.

PROOF We have two directions to prove. First, if A is decidable, we can easily see that both A and its complement $\overline{A}$ are Turing-recognizable. Any decidable language is Turing-recognizable, and the complement of a decidable language also is decidable.

Proof of Theorem 4.22 Taken from the book.

> First, if A is decidable, we can easily see that both A and its complement A are Turing-recognizable.

- A is decidable, then A is recognizable by definition.
- A is decidable, then $\overline{A}$ is recognizable? **Why?**

> Any decidable language is Turing-recognizable,

- Yes, by definition.

> and the complement of a decidable language also is decidable.

- **Why?**

If L is decidable, then $\overline{L}$ is decidable

1. Let M decide L.
2. Create a Turing machine that negates the result of M.

```
Definition inv M w :=
  mlet b ← Call m w in halt_with (negb b).
```

3. Show that `inv M` recognizes
   $$\mathrm{Inv}(L) = \{w \mid M \text{ rejects } w\}$$
4. Show that the result of `inv M` for any word w is the negation of running M with m, where negation of accept is reject, reject is accept, and loop is loop.
5. The goal is to show that `inv M` recognizes $\overline{L}$ and is decidable.

> What about loops? If M loops on some word w, then `inv M` would also loop. How is does `inv M` recognize $\overline{L}$?

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> What about loops? If M loops on some word w, then `inv M` would also loop. How is does `inv M` recognize $\overline{L}$?

Recall that L is decidable, so M will never loop.

If L is decidable, then $\overline{L}$ is decidable

Continuation...

Part 1. Show that `inv M` recognizes $\overline{L}$

We must show that: If M decides L and `inv M` recognizes $\mathbf{Inv}(L)$, then `inv M` is decidable.

It is enough to show that if M decides L, then $\mathbf{Inv}(L) = \overline{L}$.

Show proof `inv_compl_equiv`.

Part 2. Show that `inv M` is a decider

Show proof `decides_to_compl`.

Chapter 5: Undecidability

$HALT_{\mathsf{TM}}$: Termination of TM

Will this TM halt given this input?

(The Halting problem)

$HALT_{\mathsf{TM}}$ is undecidable

Theorem 5.1: HALT_TM loops for some input

Set-based encoding

$$HALT_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ halts on input } w\}$$

Function-based encoding

```
def HALT_TM(M,w):
  return M halts on w
```

Proof

Proof idea: Given Turing machine acc, show that acc decides A_{TM}.

```
def acc(M, w):
  if HALT_TM(M,w):
    return M(w)
  else:
    return False
```

$HALT_{\text{TM}}$ is undecidable

Theorem 5.1: Proof overview

```
Definition acc D p :=
  let (M, w) := decode_machine_input p in
  mlet b <- Call D p in
  if b then Call M w else REJECT.
```

```
Definition acc_lang D p :=
  let (M, w) := decode_machine_input p in
  run D p = Accept /\ run M w = Accept.
```

$\text{Acc}_D = \{\langle M, w\rangle \mid D \text{ accepts } \langle M, w\rangle \wedge M \text{ accepts } w\}$

Apply Thm 4.11 to (H) "acc decides A_{TM}" and reach a contradiction. To prove H:

1. Show that acc recognizes Acc_D
2. Show that $\text{Acc}_D = A_{TM}$ (why do we need this step?)
3. Show that acc is decidable

$HALT_{\mathsf{TM}}$ is undecidable

Part 1. Show that acc recognizes $\mathbf{Acc}_D$

```
Definition acc p :=
  let (M, w) := decode_machine_input p in
  mlet b ← Call D p in
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```

1. Show that if `acc w` accepts, then $p \in \mathbf{Acc}_D$, ie, D accepts $\langle M, p\rangle$ and M accepts w.

$HALT_{\mathsf{TM}}$ is undecidable

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1. Show that if `acc w` accepts, then $p \in \mathbf{Acc}_D$, ie, D accepts $\langle M, p\rangle$ and M accepts w.
   - Case analysis on `Call D <M,w>`

$HALT_{\mathsf{TM}}$ is undecidable

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   - Case analysis on `Call D <M,w>`
     1. `D` accepts `<M,w>`, then we get that M accepts w

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1. Show that if `acc w` accepts, then $p \in \mathbf{Acc}_D$, ie, D accepts $\langle M, p \rangle$ and M accepts w.
   - Case analysis on `Call D <M,w>`
     1. `D` accepts `<M,w>`, then we get that M accepts w
     2. `D` rejects `<M,w>`, then contradiction
2. Show that if $w \in \mathbf{Acc}_D$, then `acc w` accepts.

$HALT_{\mathsf{TM}}$ is undecidable

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Part 1. Show that acc recognizes $\mathbf{Acc}_D$

```
Definition acc p :=
  let (M, w) := decode_machine_input p in
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  if b then Call M w else REJECT.
```

1. Show that if `acc w` accepts, then $p \in \mathbf{Acc}_D$, ie, D accepts $\langle M, p \rangle$ and M accepts w.
   - Case analysis on `Call D <M,w>`
     1. `D` accepts `<M,w>`, then we get that M accepts w
     2. `D` rejects `<M,w>`, then contradiction
2. Show that if $w \in \mathbf{Acc}_D$, then `acc w` accepts.
   - Given D accepts $\langle M, w \rangle$ and M accepts w, show that `acc w` accepts

$HALT_{\mathsf{TM}}$ is undecidable

Part 1. Show that acc recognizes $\mathbf{Acc}_D$

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Definition acc p :=
  let (M, w) := decode_machine_input p in
  mlet b ← Call D p in
  if b then Call M w else REJECT.
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1. Show that if `acc w` accepts, then $p \in \mathbf{Acc}_D$, ie, D accepts $\langle M, p \rangle$ and M accepts w.
   - Case analysis on `Call D <M,w>`
     1. `D` accepts `<M,w>`, then we get that M accepts w
     2. `D` rejects `<M,w>`, then contradiction
2. Show that if $w \in \mathbf{Acc}_D$, then `acc w` accepts.
   - Given D accepts $\langle M, w \rangle$ and M accepts w, show that `acc w` accepts
   - Rewrite each in code, get accept

$HALT_{\text{TM}}$ is undecidable

Part 2. Show that $\mathbf{Acc}_D = A_{TM}$

1. Show that if $\langle M, w\rangle \in \mathbf{Acc}_D$, then $\langle M, p\rangle \in A_{TM}$

$HALT_{\mathsf{TM}}$ is undecidable

Part 2. Show that $\mathbf{Acc}_D = A_{TM}$

1. Show that if $\langle M, w\rangle \in \mathbf{Acc}_D$, then $\langle M, p\rangle \in A_{TM}$
   - We have M accepts w from $\langle M, p\rangle \in \mathbf{Acc}_D$

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   - We have M accepts w from $\langle M, p\rangle \in \mathbf{Acc}_D$
2. Show that if (i) $\langle M, w\rangle \in A_{TM}$, then $\langle M, w\rangle \in \mathbf{Acc}_D$, ie

$HALT_{\mathrm{TM}}$ is undecidable

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   - We have M accepts w from $\langle M, p\rangle \in \mathbf{Acc}_D$
2. Show that if (i) $\langle M, w\rangle \in A_{TM}$, then $\langle M, w\rangle \in \mathbf{Acc}_D$, ie M accepts w and D accepts $\langle M, w\rangle$

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   - We have that M accepts w from (i)

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1. Show that if $\langle M, w\rangle \in \mathbf{Acc}_D$, then $\langle M, p\rangle \in A_{TM}$
   - We have M accepts w from $\langle M, p\rangle \in \mathbf{Acc}_D$
2. Show that if (i) $\langle M, w\rangle \in A_{TM}$, then $\langle M, w\rangle \in \mathbf{Acc}_D$, ie M accepts w and D accepts $\langle M, w\rangle$
   - We have that M accepts w from (i)
   - We have that D accepts $\langle M, w\rangle$ since M halts.

$HALT_{\mathsf{TM}}$ is undecidable

Part 3. Show that acc is decidable

Proof by contradiction. Assume acc loops with $p = \langle M, w \rangle$ and reach a contradiction.

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Proof by contradiction. Assume acc loops with $p = \langle M, w \rangle$ and reach a contradiction.

If acc loops with p, then D accepts p and M loops with w, or D loops with p †

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Part 3. Show that acc is decidable

Proof by contradiction. Assume acc loops with $p = \langle M, w \rangle$ and reach a contradiction.

If acc loops with p, then D accepts p and M loops with w, or D loops with p [†]

- If D accepts p, then M halts with w, which contradicts with M loops with w

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Part 3. Show that acc is decidable

Proof by contradiction. Assume acc loops with $p = \langle M, w\rangle$ and reach a contradiction.

If acc loops with p, then D accepts p and M loops with w, or D loops with p [†]

- If D accepts p, then M halts with w, which contradicts with M loops with w
- If D loops with p, we reach a contradiction because D is a decider

---

[†]: Why?

E_{TM}: Emptiness of TM

(Is the language of this TM empty?)

Theorem 5.2: E_{TM} is undecidable

Set-based

$$E_{\mathsf{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$$

Function-based

```
def E_TM(M):
  return L(M) == {}
```

Proof overview: show that acc decides A_{TM}

```
def build_M1(M,w):
  def M1(x):
    if x == w:
      return M accepts w
    else:
      return False
  return M1
```

```
def acc(M, w):
  b = E_TM(build_M1(M, w))
  return not b
```

- $w \in L(\mathtt{M1}) \iff \langle \mathtt{M1} \rangle \notin E_{TM}$
- $w \in L(\mathtt{M1}) \iff w \in L(M)$

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Goal: E_{TM} decidable implies A_{TM} decidable

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Let D decide E_{TM}.

1. Show that acc recognizes A_{TM}

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Goal: E_{TM} decidable implies A_{TM} decidable

Let D decide E_{TM}.

1. Show that acc recognizes A_{TM}
   1. Show that $A_{\mathsf{TM}} = \mathrm{Acc}_D$ where $\mathrm{Acc}_D = \{\langle M, w\rangle \mid L(\mathtt{M1}_{M,w}) \neq \emptyset\}$ (`e_tm_a_tm_spec`)

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   1. Show that $A_{\mathsf{TM}} = \mathrm{Acc}_D$ where $\mathrm{Acc}_D = \{\langle M, w\rangle \mid L(\mathtt{M1}_{M,w}) \neq \emptyset\}$ (`e_tm_a_tm_spec`)
   2. Show that acc recognizes Acc_D (`E_tm_A_tm_recognizes`)

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Proof follows by contradiction.

1. Show that E_{TM} decidable implies A_{TM} decidable.
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Goal: E_{TM} decidable implies A_{TM} decidable

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   1. Show that $A_{\mathsf{TM}} = \mathrm{Acc}_D$ where $\mathrm{Acc}_D = \{\langle M, w\rangle \mid L(\mathtt{M1}_{M,w}) \neq \emptyset\}$ (`e_tm_a_tm_spec`)
   2. Show that acc recognizes Acc_D (`E_tm_A_tm_recognizes`)
2. Show that acc is a decider (`decider_E_tm_A_tm`)

Theorem 5.2: E_{TM} is undecidable

Part 1.1: Show that $A_{\mathsf{TM}} = \mathbf{Acc}_D$ where $\mathbf{Acc}_D = \{\langle M, w\rangle \mid L(\mathtt{M1}_{M,w}) \neq \emptyset\}$

Theorem `not_empty_to_accept`

1. Show that: If $L(\mathtt{M1}_{M,w}) \neq \emptyset$, then M accepts w.

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Theorem `not_empty_to_accept`

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   - Case analysis on running M with input w:
     - Case (a) M accepts w: use assumption to conclude

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Theorem `not_empty_to_accept`

1. Show that: If $L(\mathtt{M1}_{M,w}) \neq \emptyset$, then M accepts w.
   - Case analysis on running M with input w:
     - Case (a) M accepts w: use assumption to conclude
     - Case (b) M rejects w: we can conclude that $L(\mathtt{M1}_{M,w}) = \emptyset$ from (b)

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     - Case (a) M accepts w: use assumption to conclude
     - Case (b) M rejects w: we can conclude that $L(\mathtt{M1}_{M,w}) = \emptyset$ from (b)
     - Case (c) M loops with w: same as above

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Part 1.1: Show that $A_{\mathsf{TM}} = \mathbf{Acc}_D$ where $\mathbf{Acc}_D = \{\langle M, w\rangle \mid L(\mathtt{M1}_{M,w}) \neq \emptyset\}$

Theorem `accept_to_not_empty`

2. Show that: If M accepts w, then $L(\mathtt{M1}_{M,w}) \neq \emptyset$.

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Theorem `accept_to_not_empty`

2. Show that: If M accepts w, then $L(\mathtt{M1}_{M,w}) \neq \emptyset$.
    1. Proof follows by contradiction: assume $L(\mathtt{M1}_{M,w}) = \emptyset$.

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2. Show that: If M accepts w, then $L(\mathtt{M1}_{M,w}) \neq \emptyset$.
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   2. We know that $\mathtt{M1}_{M,w}$ does not accept w from (2.1)

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    3. To contradict 2.2, we show that $\mathtt{M1}_{M,w}$ **accepts** w

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    2. We know that $\mathtt{M1}_{M,w}$ does not accept w from (2.1)
    3. To contradict 2.2, we show that $\mathtt{M1}_{M,w}$ **accepts** w
        1. Since $x = w$ and (2.1), then $\mathtt{M1}_{M,w}$ **accepts** w