Today we will learn...

Decidability of
- The Halting Problem
- Emptiness for TM
- Regularity
- Equality

Section 5.1
Decidable languages:

- \( A_{DFA}, A_{REX}, A_{NFA}, A_{CFG} \)

\[
A_{DFA} = \{ \langle D, w \rangle \mid D \text{ accepts } w \}
\]

- \( E_{DFA}, E_{CFG} \)

\[
E_{DFA} = \{ \langle D \rangle \mid L(D) = \emptyset \}
\]

- \( EQ_{DFA} \)

\[
EQ_{DFA} = \{ \langle N_1, N_2 \rangle \mid L(N_1) = L(N_2) \}
\]
Exercise 1

Prove or falsify the following statement: $EQ_{REX}$ is undecidable.
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Prove or falsify the following statement: $EQ_{REX}$ is undecidable.

**Proof.** False. $EQ_{REX}$ is decidable, as given by the following pseudo code, where $EQ_{DFA}$ is the decider of $EQ_{DFA}$ and $REX_{TO_{DFA}}$ is the conversion from a regular expression into a DFA.

```python
def EQ_REX(R1, R2):
    return EQ_DFA(REX_TO_DFA(R1), REX_TO_DFA(R2))
```
Exercise 2

Let $D$ be the DFA below

![DFA Diagram]

- Exercise 2.1: Is $\langle D, 0100 \rangle \in A_{DFA}$?
- Exercise 2.2: Is $\langle D, 101 \rangle \in A_{DFA}$?
- Exercise 2.3: Is $\langle D \rangle \in A_{DFA}$?
- Exercise 2.4: Is $\langle D, 101 \rangle \in A_{REX}$?
- Exercise 2.5: Is $\langle D \rangle \in E_{DFA}$?
- Exercise 2.6: Is $\langle D, D \rangle \in EQ_{DFA}$?
- Exercise 2.7: Is $101 \in A_{REX}$?

```python
def A_DFA(D, w):
    return D.accept(w)
def E_DFA(D):
    return L(D) == {}  
def EQ_DFA(D1, D2):
    return L(D1) == L(D2)
```
Exercise 3

Recall that DFAs are closed under $\cap$. Prove the following statement.

If $A$ is regular, then $X_A$ decidable.

$$X_A = \{ \langle D \rangle \mid D \text{ is a DFA} \land L(D) \cap A \neq \emptyset \}$$
Exercise 3

Recall that DFAs are closed under $\cap$. Prove the following statement.

If $A$ is regular, then $X_A$ decidable.

$$X_A = \{ \langle D \rangle \mid D \text{ is a DFA } \land L(D) \cap A \neq \emptyset \}$$

**Proof.** If $A$ is regular, then let $C$ be the DFA that recognizes $A$. Let intersect be the implementation of $\cap$ and E_DFA the decider of $E_{DFA}$. The following is the decider of $X_A$.

```
def X_A(D):
    return not E_DFA(intersect(C, D))
```
Theorem 4.22

$L$ decidable iff $L$ recognizable and $L$ co-recognizable
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$L$ decidable iff $L$ recognizable and $L$ co-recognizable

**Proof.** We can divide the above theorem in the following three results.

1. If $L$ decidable, then $L$ is recognizable. (**Proved.**)
2. If $L$ decidable, then $L$ is co-recognizable. (**Proved.**)
3. If $L$ recognizable and $L$ co-recognizable, then $L$ decidable.
Part 3. If $L$ recognizable and $\overline{L}$ recognizable, then $L$ decidable.

We need to extend our mini-language of TMs

```plaintext
plet b ← P1 \ P2 in P3
Runs P1 and P2 in parallel.
  • If P1 and P2 loop, the whole computation loops
  • If P1 halts and P2 halts, pass the success of both to P3
  • If P1 halts and P2 loops, pass the success of P1 to P3
  • If P1 loops and P2 halts, pass the success of P2 to P3
```

```
Inductive par_result :=
| pleft: bool → par_result
| pright: bool → par_result
| pboth: bool → bool → par_result.
```
Part 3. If \( L \) recognizable and \( \overline{L} \) recognizable, then \( L \) decidable.

**Proof.**

1. Let \( M_1 \) recognize \( L \) from assumption \( L \) recognizable
2. Let \( M_2 \) recognize \( \overline{L} \) from assumption \( \overline{L} \) recognizable
3. Build the following machine

\[
\begin{align*}
\text{Definition} \quad & \text{par\_run} \ M_1 \ M_2 \ w := \\
& \begin{array}{l}
\text{plet} \ b \leftarrow \text{Call} \ M_1 \ w \parallel \text{Call} \ M_2 \ w \ \text{in} \\
\text{match} \ b \ \text{with} \\
| \ \text{pleft} \ \text{true} & \Rightarrow \text{ACCEPT} \\
| \ \text{pboth} \ \text{true} & \Rightarrow \text{ACCEPT} \\
| \ _ & \Rightarrow \text{REJECT}
\end{array} \\
\text{end.}
\end{align*}
\]

(* \( M_1 \) and \( M_2 \) are parameters of the machine *)

(* Call \( M_1 \) with \( w \) and \( M_2 \) with \( w \) in parallel *)

(* If \( M_1 \) accepts \( w \), accept *)

(* Otherwise, reject *)

4. Show that \( \text{par\_run} \ M_1 \ M_2 \) recognizes \( L \) and is a decider.
Part 3. If $L$ recognizable and $\overline{L}$ recognizable, then $L$ decidable.

Point 4: Show that $\text{par\_run} \ M1 \ M2$ recognizes $L$ and is a decider.

- 1. Show that $\text{par\_run} \ M1 \ M2$ recognizes $L$: $\text{par\_run} \ M1 \ M2$ accepts $w$ iff $L(w)$
- 1.1. $\text{par\_run} \ M1 \ M2$ accepts $w$, then $w \in L$
- 1.2. $w \in L$, then $\text{par\_run} \ M1 \ M2$ accepts $w$ case analysis on run $M2$ with $w$

**Definition** $\text{par\_run} \ M1 \ M2 \ w :=$

\[
\text{plet } b \leftarrow \text{Call } M1 \ w \ \text{\textbackslash\textbackslash } \text{Call } M2 \ w \text{ in}
\text{match } b \text{ with}
| \text{pleft } \text{true} \Rightarrow \text{ACCEPT}
| \text{pboth } \text{true } \_ \Rightarrow \text{ACCEPT}
| \_ \Rightarrow \text{REJECT}
\text{end}.
\]

- $M1$ recognizes $L$
- $M2$ recognizes $\overline{L}$
- Lemma $\text{par\_mach\_lang}$
Part 3. If $L$ recognizable and $\overline{L}$ recognizable, then $L$ decidable.

Point 4: Show that $\text{par\_run \ M1 \ M2}$ recognizes $L$ and is a decider.

1. Show that $\text{par\_run \ M1 \ M2}$ recognizes $L$: $\text{par\_run \ M1 \ M2}$ accepts $w$ iff $L(w)$
   1. $\text{par\_run \ M1 \ M2}$ accepts $w$, then $w \in L$ by case analysis on $\text{Call M1 } w \setminus \setminus \text{Call M2 } w$:
      - $\text{pleft \ true and M1 accepts } w$: holds since $M1$ recognizes $L$
      - $\text{both \ true }$ and $M1$ accepts $w$: same as above
      - otherwise: contradiction
   2. $w \in L$, then $\text{par\_run \ M1 \ M2}$ accepts $w$ case analysis on run $M2$ with $w$
      - $M2$ accept $w$: $\text{par\_run \ M1 \ M2}$ accept since $M1$ accepts with $w$
      - $M2$ loops $w$: $\text{par\_run \ M1 \ M2}$ accept since $M1$ accepts with $w$
      - $M2$ reject $w$: $\text{par\_run \ M1 \ M2}$ accept since $M1$ accepts with $w$
Part 3. If $L$ recognizable and $\overline{L}$ recognizable, then $L$ decidable.

Point 4: Show that $\text{par\_run } M_1 \ M_2$ recognizes $L$ and is a decider.

2. Show that $\text{par\_run } M_1 \ M_2$ decides $L$

(Walk through the proof of \text{recognizable\_co\_recognizable\_to\_decidable}...)