CS420
Introduction to the Theory of Computation
Lecture 4: Nondeterministic Finite Automaton
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Revisiting what we learned...

- Operations on words; set theory
- How to draw a state diagram from a DFA?
- A step-by-step union example
- Reduction graphs with $\epsilon$-transitions?
- What is the powerset function?
- How to draw a state diagram from an NFA?
- How to convert an NFA into a DFA?
HW1 heads up
HW1 heads up

- Answers should all be given as state diagrams
- Simplification of $L_5$:
  - If the resulting DFA is already simplified, then just answer "the same DFA"
  - We have no way of proving that the DFA is the smallest
- After applying the union operator you should not simplify the final diagram
- When writing down a diagram from an $M$ (directly using a transition function) you should not simplify it
Operations on words; set theory
What is $w^n; L^*$
Operations on words; set theory

What is $w^n; L^*$

**Answer:** (Lecture 1, slides 38 and 39) concatenate $w$ with itself $n$ times. $L^* = \{w^n \mid w \in L \land n \geq 0\}$ (Lecture 2; slide 19)

See also **Definition 1.23** in the book.
Operations on words; set theory

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What is $w_1, w, w_2 \in s$?
Operations on words; set theory

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What is $w_1, w, w_2 \in s$?

**Answer:** Shorthand notation for $w_1 \in s \land w \in s \land w_2 \in s$. 
Operations on words; set theory

What is \( w^n; L^* \)

**Answer:** (Lecture 1, slides 38 and 39) concatenate \( w \) with itself \( n \) times. \( L^* = \{ w^n \mid w \in L \land n \geq 0 \} \) (Lecture 2; slide 19)

See also **Definition 1.23** in the book.

What is \( w_1, w, w_2 \in s? \)

**Answer:** Shorthand notation for \( w_1 \in s \land w \in s \land w_2 \in s \).

What is \( \{ w \mid P(w) \} \)?

**Answer:** this is known as the **set-builder notation** (set comprehension). I assume you learned this in CS220 (or prior). It is a way of saying any \( w \) such that \( P(x) \) holds. For instance, \( \{ w \mid |w| \text{ is even} \land w = ab \cdot w_2 \land w_2 \in \Sigma^* \} \) means that \( w \) is such that: \( |w| \) is even (the length of \( w \) is even) **AND** \( w = ab \cdot w_2 \) (\( w \) starts with \( ab \) followed by \( w_2 \), **AND** \( w_2 \in \Sigma^* \) (\( w_2 \) is a word).
How to draw a state diagram for a DFA
How to draw a state diagram for a DFA

Give the DFA of $M = (\{q_1, q_2, q_3, q_4, q_5\}, \{a, b\}, \delta, q_1, \{q_4\})$

where

$\delta(q_1, a) = q_2$
$\delta(q_1, b) = q_3$
$\delta(q_2, a) = q_1$
$\delta(q_2, b) = q_3$
$\delta(q_3, a) = q_5$
$\delta(q_4, a) = q_1$
$\delta(q_4, b) = q_2$
$\delta(q, c) = q$ otherwise

1. pick $q_1$; draw edge for each $\Sigma$, one for $a$; another for $b$
How to draw a state diagram for a DFA

Give the DFA of $M = (\{q_1, q_2, q_3, q_4, q_5\}, \{a, b\}, \delta, q_1, \{q_4\})$ where

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$\delta(q_3, a) = q_5$
$\delta(q_4, a) = q_1$
$\delta(q_4, b) = q_2$
$\delta(q, c) = q$ otherwise

1. pick $q_1$; draw outgoing edges
2. pick $q_2$ (or $q_3$); draw outgoing edges
How to draw a state diagram for a DFA

Give the DFA of $M = (\{q_1, q_2, q_3, q_4, q_5\}, \{a, b\}, \delta, q_1, \{q_4\})$

where

$$\delta(q_1, a) = q_2$$
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$$\delta(q_3, a) = q_5$$
$$\delta(q_4, a) = q_1$$
$$\delta(q_4, b) = q_2$$
$$\delta(q, c) = q \text{ otherwise}$$

1. pick $q_1$; draw outgoing edges
2. pick $q_2$; draw outgoing edges
3. pick $q_3$; draw outgoing edges
How to draw a state diagram for a DFA

Give the DFA of $M = (\{q_1, q_2, q_3, q_4, q_5\}, \{a, b, c\}, \delta, q_1, \{q_4\})$

where

\[
\begin{align*}
\delta(q_1, a) &= q_2 \\
\delta(q_1, b) &= q_3 \\
\delta(q_2, a) &= q_1 \\
\delta(q_2, b) &= q_3 \\
\delta(q_3, a) &= q_5 \\
\delta(q_4, a) &= q_1 \\
\delta(q_4, b) &= q_2 \\
\delta(q, c) &= q \text{ otherwise}
\end{align*}
\]

1. pick $q_1$; draw outgoing edges
2. pick $q_2$; draw outgoing edges
3. pick $q_3$; draw outgoing edges
4. pick $q_5$; draw outgoing edges
How to draw a state diagram for a DFA

Give the DFA of \( M = (\{q_1, q_2, q_3, q_4, q_5\}, \{a, b\}, \delta, q_1, \{q_4\}) \) where

\[
\begin{align*}
\delta(q_1, a) &= q_2 \\
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\delta(q_4, b) &= q_2 \\
\delta(q, c) &= q \text{ otherwise}
\end{align*}
\]

1. pick \( q_1 \); draw outgoing edges
2. pick \( q_2 \); draw outgoing edges
3. pick \( q_3 \); draw outgoing edges
4. pick \( q_5 \); draw outgoing edges

**Note 1:** state \( q_4 \) is not present in our graph, because it is unreachable. We only render reachable states in our state diagrams.

**Note 2:** do not attempt to simplify the DFA.
Step-by-step union example
Step-by-step union example

We start from the pair \((q_1, q_1)\) (the initial state of each DFA) which we denote by \(q_{1,1}\). For each element of \(\Sigma\) draw an edge.
Step-by-step union example

At $q_{1,1}$

- Read 0. (Left) From $q_1$ we advance to $q_2$. (Right) From $q_1$ we advance to $q_4$. Result $q_{2,4}$
- Read 1. (Left) From $q_1$ we advance to $q_1$. (Right) From $q_1$ we advance to $q_2$. Result $q_{1,2}$. 
Step-by-step union example

At \( q_{2,4} \):
- Read 0. (Left) From \( q_2 \) we advance to \( q_2 \). (Right) From \( q_4 \) we advance to \( q_4 \). Result \( q_{2,4} \).
- Read 1. (Left) From \( q_2 \) we advance to \( q_3 \). (Right) From \( q_4 \) we advance to \( q_4 \). Result \( q_{3,4} \).
Step-by-step union example

At $q_{3,4}$:
- Read 0. (Left) From $q_3$ we advance to $q_2$. (Right) From $q_4$ we advance to $q_4$. Result $q_{2,4}$
- Read 1. (Left) From $q_3$ we advance to $q_1$. (Right) From $q_4$ we advance to $q_4$. Result $q_{1,4}$.
Step-by-step union example

At $q_{1,4}$:

- Read 0. (Left) From $q_1$ we advance to $q_2$. (Right) From $q_4$ we advance to $q_4$. Result $q_{2,4}$
- Read 1. (Left) From $q_1$ we advance to $q_1$. (Right) From $q_4$ we advance to $q_4$. Result $q_{1,4}$. 
Step-by-step union example

At $q_{1,2}$:
- Read 0. (Left) From $q_1$ we advance to $q_2$. (Right) From $q_2$ we advance to $q_3$. Result $q_{2,3}$
- Read 1. (Left) From $q_1$ we advance to $q_1$. (Right) From $q_2$ we advance to $q_4$. Result $q_{1,4}$. 
Step-by-step union example

- Read $\emptyset$. (Left) From $q_2$ we advance to $q_2$. (Right) From $q_3$ we advance to $q_3$. Result $q_{2,3}$
- Read 1. (Left) From $q_2$ we advance to $q_3$. (Right) From $q_3$ we advance to $q_3$. Result $q_{3,3}$. 
Step-by-step union example

- Read $\emptyset$. (Left) From $q_3$ we advance to $q_2$. (Right) From $q_3$ we advance to $q_3$. Result $q_{2,3}$
- Read 1. (Left) From $q_3$ we advance to $q_1$. (Right) From $q_3$ we advance to $q_3$. Result $q_{1,3}$. 
Step-by-step union example

- Read 0. (Left) From $q_1$ we advance to $q_2$. (Right) From $q_3$ we advance to $q_3$. Result $q_{2,3}$
- Read 1. (Left) From $q_1$ we advance to $q_1$. (Right) From $q_3$ we advance to $q_3$. Result $q_{1,3}$. 
Step-by-step union example

**Note:** in the HW/mini-tests do **not** attempt to simplify the resulting DFA unless explicitly requested to do so.
Reduction graphs
with $\epsilon$-transitions
Reduction graphs with $\epsilon$-transitions

Acceptance for $ba$: epsilon-step

$q_1$

$q_2$

$q_3$
Reduction graphs with $\epsilon$-transitions

Acceptance for $ba$: input-step $b$

Note: at this point that are two concurrent states: $q_1$ and $q_2$, so we can consume $b$ from either (although we can only do so via $q_1$).
Reduction graphs with $\epsilon$-transitions

Acceptance for $ba$: epsilon-step
Reduction graphs with $\epsilon$-transitions

Acceptance for $ba$: input-step
Reduction graphs with $\epsilon$-transitions

Acceptance for ba: epsilon-step
Reduction graphs with $\epsilon$-transitions

Acceptance for $ba$

\[
\begin{align*}
q_1 &\xrightarrow{\epsilon} q_2 \\
q_1 &\xrightarrow{b} q_3 \\
q_2 &\xrightarrow{\epsilon} q_1 \\
q_2 &\xrightarrow{a} q_3 \\
q_3 &\xrightarrow{\epsilon} q_2
\end{align*}
\]
What is the Powerset function?
What is the Powerset function?

Given a set it returns a set that consists of all possible subsets of that set and itself.

\[ \mathcal{P}(s) = \{ r \mid r \subseteq s \} \]

Example

\[ \mathcal{P}(\{q_1, q_2, q_3\}) = \]
What is the Powerset function?

Given a set it returns a set that consists of all possible subsets of that set and itself.

\[ P(s) = \{ r \mid r \subseteq s \} \]

Example

\[ P(\{q_1, q_2, q_3\}) = \{\emptyset, \{q_1\}, \{q_2\}, \{q_3\}, \{q_1, q_2\}, \{q_1, q_3\}, \{q_2, q_3\}, \{q_1, q_2, q_3\}\} \]
How to draw a state diagram for an NFA
How to draw a state diagram for an NFA

\[ M = (\{q_1, q_2, q_3, q_4\}, \{a, b\}, \delta, q_1, \{q_2, q_4\}) \]

where

\[
\begin{align*}
\delta(q_1, a) &= \{q_1, q_2, q_3\} \\
\delta(q_1, \epsilon) &= \{q_3\} \\
\delta(q_2, a) &= \{q_1, q_3\} \\
\delta(q_2, b) &= \{q_2\} \\
\delta(q_3, b) &= \{q_4\} \\
\delta(q_3, \epsilon) &= \{q_1\} \\
\delta(q, c) &= \emptyset \text{ otherwise}
\end{align*}
\]
How to draw a state diagram for an NFA

\[ M = (\{q_1, q_2, q_3, q_4\}, \{a, b\}, \delta, q_1, \{q_2, q_4\}) \]

where

\[
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\delta(q_1, \varepsilon) &= \{q_3\} \\
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\delta(q, c) &= \emptyset \text{ otherwise}
\end{align*}
\]
Producing a DFA from an NFA
Producing a DFA from an NFA

The initial state is the set of all states in the NFA that are reachable from $q_1$ via $\epsilon$ transitions plus $q_1$. 
Producing a DFA from an NFA

For each input in $\Sigma$ range we must draw a transition to a target state.
A target state is found by taking an input, say $a$, and doing an input+epsilon step on each sub-state.
Producing a DFA from an NFA

First, input a we find all reachable states (via input+epsilon state) that start from either $q_1$ or $q_3$.

- From $q_1$ via $a$ we get $\{q_1, q_2, q_3\}$
- From $q_3$ via $a$ we get $\emptyset$
- Result state is $\{q_1, q_2, q_3\} \cup \emptyset = \{q_1, q_2, q_3\}$
Producing a DFA from an NFA

Second, input $b$ we find all reachable states (via input+epsilon state) that start from either $q_1$ or $q_3$.

- From $q_1$ via $b$ we get $\emptyset$
- From $q_3$ via $b$ we get $\{q_4\}$
- Result state is $\emptyset \cup \{q_4\} = \{q_4\}$
Producing a DFA from an NFA

For inputs \( a \) and \( b \) we find all reachable states (via input + epsilon state) that start from \( q_4 \):

- From \( q_4 \) via \( a \) we get \( \emptyset \), so the result state is \( \emptyset \)
- From \( q_4 \) via \( b \) we get \( \emptyset \), so the result state is \( \emptyset \)
Producing a DFA from an NFA

Transition from \( \{ q_1, q_2, q_3 \} \) via \( a \)?

- We know with \( \{ q_1, q_3 \} \) with \( a \) we reach \( \{ q_1, q_2, q_3 \} \)
- From \( q_2 \) with \( a \) we reach \( \{ q_3 \} \)
- Thus, result state is \( \{ q_1, q_2, q_3 \} \cup \{ q_3 \} = \{ q_1, q_2, q_3 \} \) (self-loop)
Producing a DFA from an NFA

Transition from \( \{q_1, q_2, q_3\} \) via \( b \)?

- We know with \( \{q_1, q_3\} \) with \( b \) we reach \( \{q_4\} \)
- From \( q_2 \) with \( b \) we reach \( \{q_2\} \)
- Thus, result state is \( \{q_4\} \cup \{q_2\} = \{q_2, q_4\} \)
Producing a DFA from an NFA

Transition from $\{q_2, q_4\}$ via $a$?
- From $q_2$ with $a$ we reach $\{q_1, q_3\}$
- From $q_4$ with $a$ we reach $\emptyset$
- Thus, result state is $\{q_1, q_3\} \cup \emptyset = \{q_1, q_3\}$
Producing a DFA from an NFA

Transition from \( \{q_2, q_4\} \) via \( b \):
- From \( q_2 \) with \( b \) we reach \( \{q_2\} \)
- From \( q_4 \) with \( b \) we reach \( \emptyset \)
- Thus, result state is \( \{q_2\} \cup \emptyset = \{q_2\} \)
Transition from \( \{q_2\} \) via \( a \)?
- From \( q_2 \) with \( a \) we reach \( \{q_1, q_3\} \) (result state)
Producing a DFA from an NFA

Transition from \( \{q_2\} \) via \( b \)?
- From \( q_2 \) with \( b \) we reach \( \{q_2\} \) (result state; self loop)
State \{\} (also known as \emptyset) is a sink state, so we draw a self loop for every input in \Sigma.
Today we will learn...

- Nil
- Empty
- Character
- Union
- Concatenation
- Star

Section 1.2
The \( \text{nil}_\Sigma \) operator

\[ L(\text{nil}_\Sigma) = \emptyset \]
The $\text{nil}_\Sigma$ operator

$L(\text{nil}_\Sigma) = \emptyset$

Example $\Sigma = \{a, b\}$
The nil\_\Sigma operator

\[ L(\text{nil}_\Sigma) = \emptyset \]

Example \( \Sigma = \{a, b\} \)

Implementation

```python
def make_nil(alphabet):
    Q1 = 0
    return NFA(
        states=[Q1], alphabet=alphabet, transition_func=lambda q, a: {},
        start_state=Q1, accepted_states=[])
The empty$_{\Sigma}$ operator

$L(\text{empty}_\Sigma) = \{\epsilon\}$
The empty$_\Sigma$ operator

$L(\text{empty}_\Sigma) = \{\epsilon\}$

Example $\Sigma = \{a, b\}$
The empty $\Sigma$ operator

$L(\text{empty}_\Sigma) = \{\epsilon\}$

Example $\Sigma = \{a, b\}$

Implementation

```python
def make_empty(cls, alphabet):
    Q1 = 0
    return NFA(
        states = [Q1],
        alphabet = alphabet,
        transition_func = lambda q, a: {},
        start_state = Q1, accepted_states = [Q1])
```
The $\text{char}_{\Sigma}(c)$ operator

$L(\text{empty}_{\Sigma}(c)) = \{[c]\}$
The \( \text{char}_\Sigma(c) \) operator

\[
L(\text{empty}_\Sigma(c)) = \{ [c] \}
\]

Example \( \Sigma = \{ a, b \} \)
The \texttt{char}_\Sigma(c) \operatorname{operator}

\[ L(\text{empty}_\Sigma(c)) = \{ [c] \} \]

Example \( \Sigma = \{ a, b \} \)

![Diagram](image)

Implementation

```python
def make_char(cls, alphabet, char):
    states = Q1, Q2 = 0, 1
    def transition(q, a):
        return {Q2} if a == char and q == Q1 else {}
    return cls(states, alphabet, transition, Q1, [Q2])
```
The \textbf{union}(M, N) operator

\[ L(\text{union}(M, N)) = L(M) \cup L(N) \]
The \textbf{union}(M, N) \textbf{operator}

\[ L(\text{union}(M, N)) = L(M) \cup L(N) \]

\[ N_1 \]

\[ N_2 \]

\[ \text{union}(N_1, N_2) =? \]
The \textbf{union}(M, N) operator

\[ L(\text{union}(M, N)) = L(M) \cup L(N) \]

Example \textbf{union}(N_1, N_2)

- Add a new initial state
- Connect new initial state to the initial states of \( N_1 \) and \( N_2 \) via \( \epsilon \)-transitions.
The \texttt{concat}(M, N) operator

\[
L(\text{concat}(M, N)) = L(M) \cdot L(N)
\]
The `concat(M, N)` operator

\[ L(\text{concat}(M, N)) = L(M) \cdot L(N) \]

Example 1: \( L(\text{concat}([a], [b])) = \{ab\} \)
The **concat**($M$, $N$) operator

$L(\text{concat}(M, N)) = L(M) \cdot L(N)$

**Example 1:** $L(\text{concat}(\text{char}(a), \text{char}(b))) = \{ab\}$

**Solution**

**What did we do?** Connect the accepted states of $N_1$ to the initial state of $N_2$ via $\epsilon$-transitions.

Why not connect directly from $q_{1,1}$ into $q_{1,2}$? See next slide.
Concatenation example 2

Solution

\[
\begin{aligned}
\text{start} & \rightarrow q_{1,1} \rightarrow q_{2,1} \rightarrow q_{1,2} \rightarrow q_{2,2} \\
q_1 & \rightarrow q_2 \quad a \\
q_2 & \rightarrow q_3 \quad a \\
q_3 & \rightarrow q_4 \quad \epsilon, a, \epsilon, a \\
q_4 & \rightarrow q_2 \quad b
\end{aligned}
\]
Concatenation example 2

Solution
Concatenate two NFAs

Let $N_1 = (Q_1, \Sigma_1, \delta_1, q_1, F_1)$, $N_2 = (Q_2, \Sigma_2, \delta_2, q_2, F_2)$, $\text{tag}(Q, n) = \{q^n \mid q \in Q\}$. We have that $N_1 \cdot N_2 = (Q, \Sigma, \delta, q_1^1, F_2)$ where:

- $Q = \text{tag}(Q_1, 1) \cup \text{tag}(Q_2, 2)$
- $\Sigma = \Sigma_1 \cup \Sigma_2$
- $\delta(q^1, \epsilon) = \{q_2^2\}$ if $q \in F_1$ (Note: $q_2^2$ represents the starting state of $N_2$ tagged with 2.)
- $\delta(q^n, a) = \text{tag}(\delta_n(r, a), n)$ if $n \in \{1, 2\}$
The \textbf{star}($N$) operator

\[ L(\text{star}(N)) = L(N)^* \]
The \texttt{star}($N$) operator

$L(\texttt{star}(N)) = L(N)^*$

Example: $L(\texttt{star} \texttt{concat} \texttt{char}(a), \texttt{char}(b))) = \{w \mid w \text{ is a sequence of ab or empty}\}$

Solution
The \textbf{star}(N) operator

\[ L(\text{star}(N)) = L(N)^* \]

Example: \( L(\text{star}(\text{concat(char(a), char(b)))))) = \{w \mid w \text{ is a sequence of } ab \text{ or empty} \} \)

Solution

- create a new state \( q_{1,1} \)
- \( \epsilon \)-transitions from \( q_{1,1} \) to initial state
- \( \epsilon \)-transitions from accepted states to \( q_{1,1} \)
- \( q_{1,1} \) is the only accepted state
The \textbf{star}(N) operator

\[ L(\text{star}(N)) = L(N)^* \]
The \textbf{star}(N) operator

\[ L(\text{star}(N)) = L(N)^* \]