CS420

Introduction to the Theory of Computation

Lecture 2: An algebra of automatons

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Today we will introduce...

- Standard operations on languages (union, concatenation, exponentiation, kleene star)
- The nil automaton
- The empty automaton
- The character automaton
- The union automaton

Section 1.1

Formal definition of a Finite Automaton

Definition 1.5

A finite automaton is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where

1. Q is a finite set called states
2. Σ is a finite set called alphabet
3. $\delta \colon Q \times \Sigma \to Q$ is the transition function
   (δ takes a state and an alphabet and produces a state)
4. $q_0 \in Q$ is the start state
5. $F \subseteq Q$ is the set of accepted states

> A formal definition is a precise mathematical language. In this example, item declares a name and possibly some constraint, e.g., $q_0 \in Q$ is saying that q_0 **must** be in set Q. These constraints are visible in the code in the form of assertions.

Formal declaration of our running example

Let the running example be the following finite automaton $M_{turnstile}$

$$(\{\mathtt{Open}, \mathtt{Close}\}, \{\mathtt{Neither}, \mathtt{Front}, \mathtt{Rear}, \mathtt{Both}\}, \delta, \mathtt{Close}, \{\mathtt{Close}\})$$

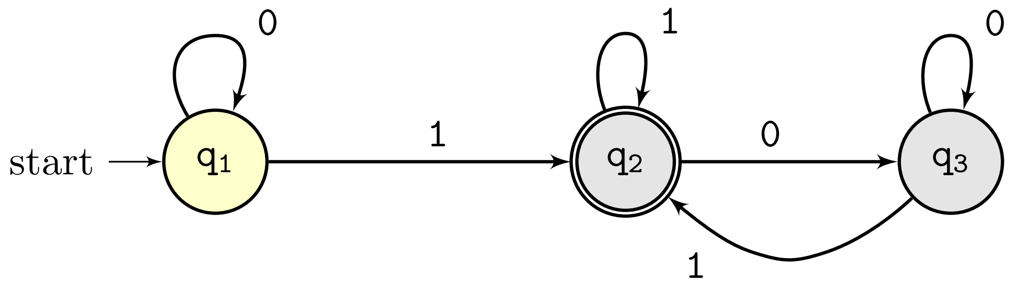
where

$$\begin{aligned} \delta(\mathtt{Close}, \mathtt{Front}) &= \mathtt{Open} \\ \delta(\mathtt{Open}, \mathtt{Neither}) &= \mathtt{Close} \\ \delta(q, i) &= q \end{aligned}$$

Facts

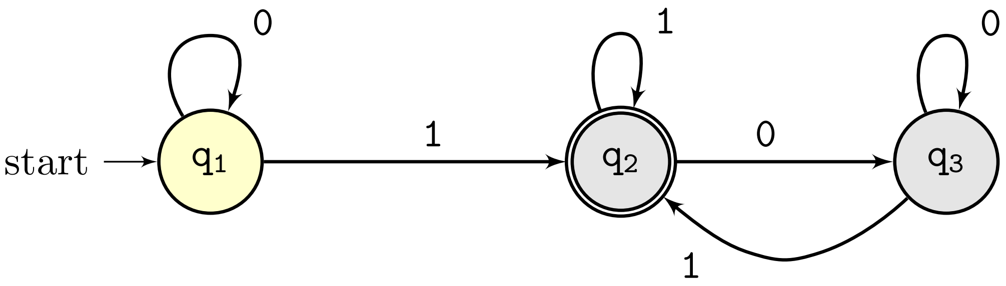
- $M_{turnstile}$ accepts [Front, Neither]
- $M_{turnstile}$ rejects [Rear, Front, Front]
- $M_{turnstile}$ accepts [Rear, Front, Rear, Neither, Rear]

Example



States?

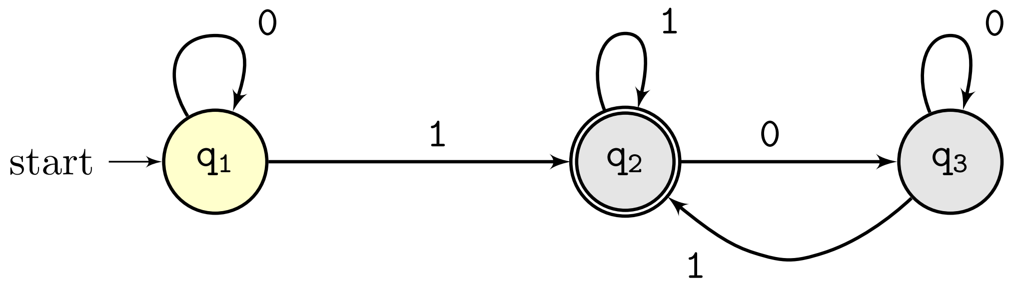
Example



States? $Q = \{q_1, q_2, q_3\}$

Alphabet?

Example

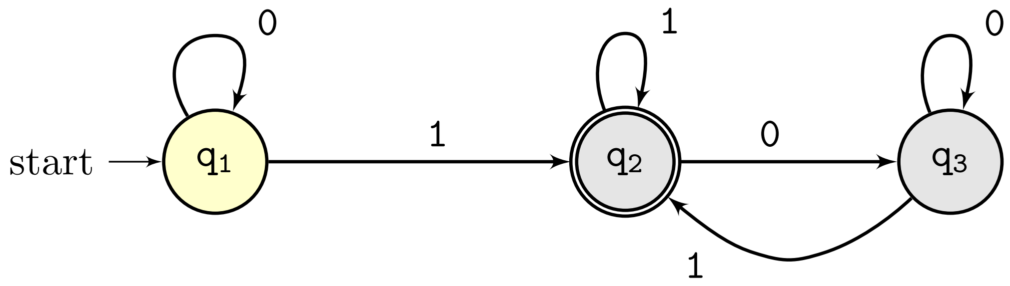


States? $Q = \{q_1, q_2, q_3\}$

Alphabet? $\Sigma = \{0, 1\}$

Transition table δ?

Example



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(prev)	***0***	***1***
q_1	q_1	q_2
q_2	q_3	q_2
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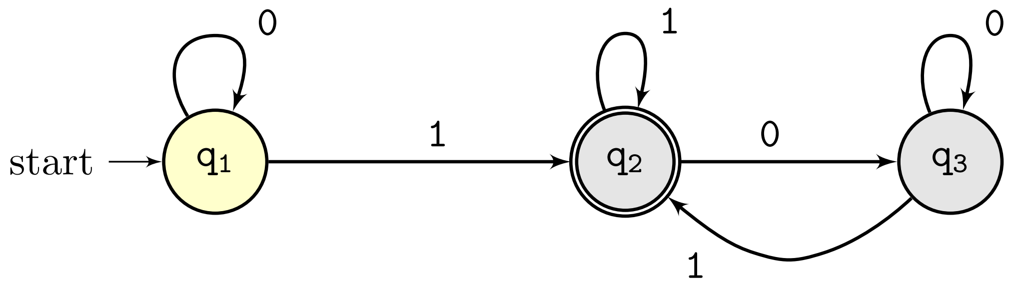

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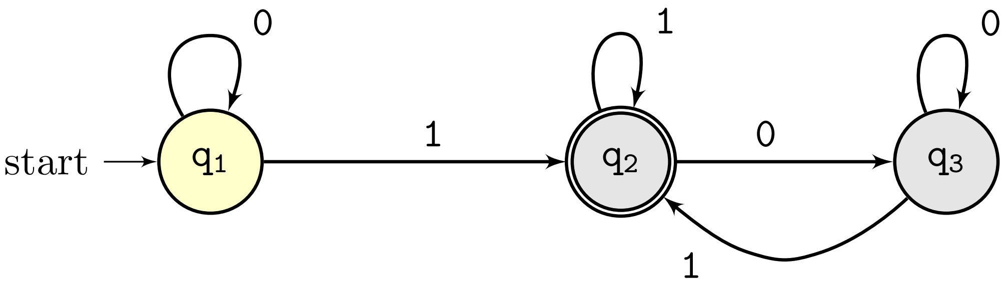
Finite Automaton:

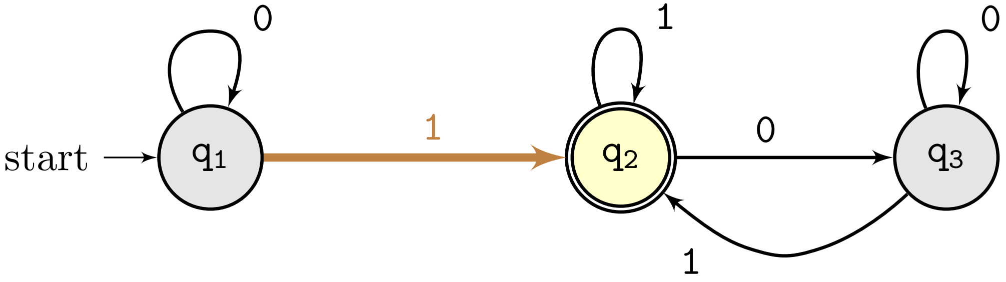
$$(\{q_1, q_2, q_3\}, \{0, 1\}, q_1, \{q_2\})$$

Example



[1, 0, 1, 1]

Example



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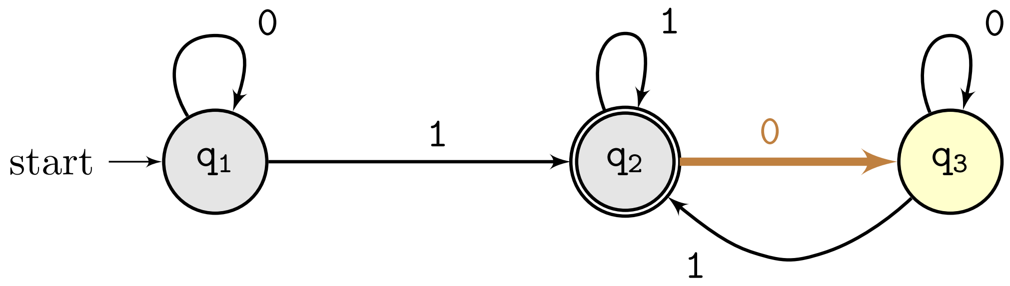
Example



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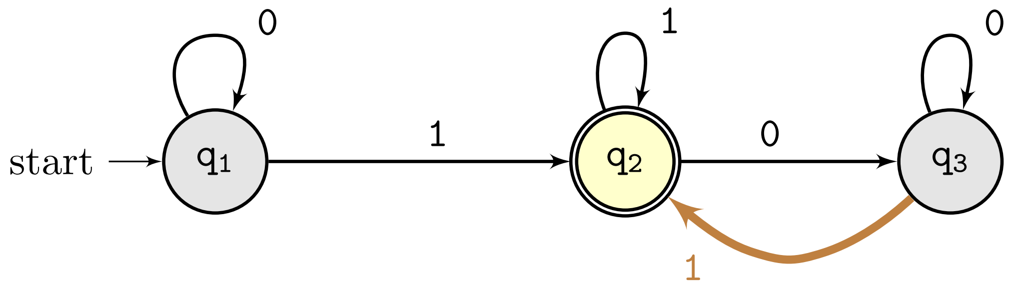

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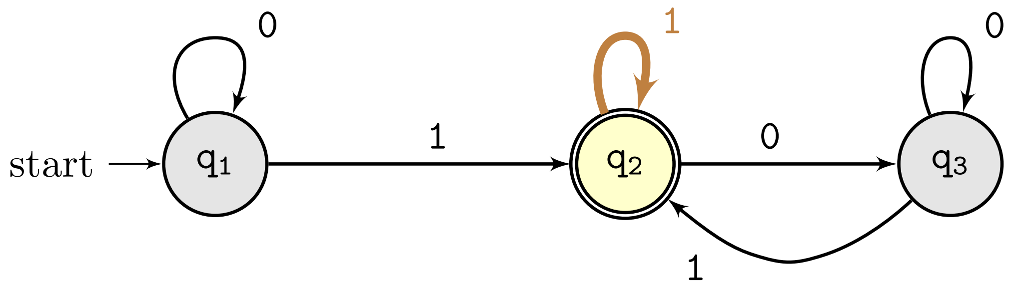

[1, 0, 1, **1**]

What are the set of inputs accepted by this automaton?

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Answer: Strings terminating in 1

The language of a machine

Definition: language of a machine

1. We define $\mathrm{L}(M)$ to be the set of all strings accepted by finite automaton M.
2. Let $A = \mathrm{L}(M)$, we say that the finite automaton M **recognizes** the set of strings A.

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Notes

- The language is the *set* of all possible alphabet-sequences recognized by a finite automaton
- Since $\mathrm{L}(M)$ is a **total** function, then the language recognized by a machine always exists and is unique
- A language may be empty
- We **cannot** write a program that returns the language of an arbitrary finite automaton. Why? **Because the language set may be infinite. How could a program return $\Sigma^\star$?**

A total function is defined for all inputs.

From a language to an automaton

Define a finite automaton that recognizes `{[h,i], [h,o]}`

- What is the alphabet of the finite automaton?

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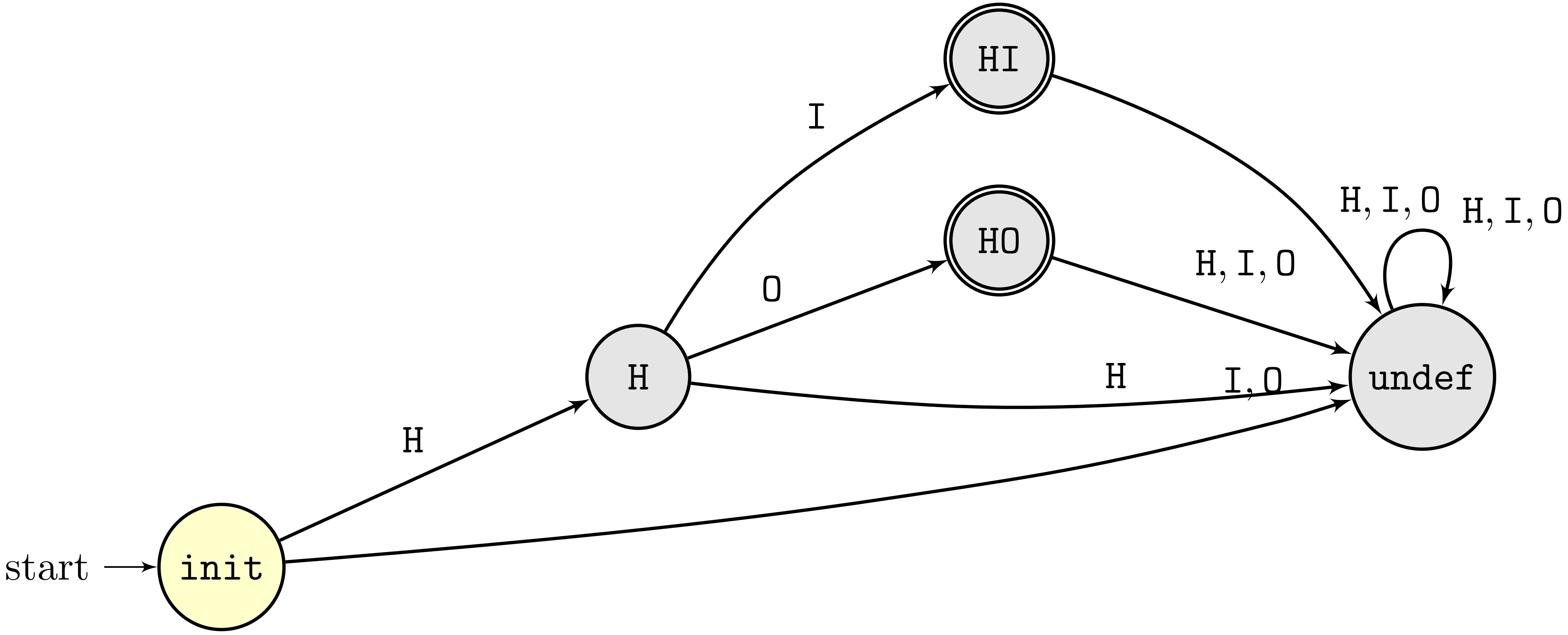
- What are the accepted states? $\{q_2, q_3\}$
- What happens if we read h in q_3? We need a "reject" state, say q_5, that every unexpected letter takes us to.

Example



Standard operations on languages

Standard operations on languages

1. union (since a language is a set of strings, we can use the union of two languages)
2. concatenation
3. the Kleene star

Concatennation

- $L_1 \cdot L_2 = \{w_1 \cdot w_2 \mid w_1 \in L_1 \wedge w_2 \in L_2\}$

Examples

1. $\{a, aa\} \cdot \{b, bb\} =$

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4. $\{a, aa, aaa\} \cdot \{\epsilon\} = \{a, aa, aaa\}$

Exponentiation

- $L^0 = \{\epsilon\}$
- $L^{n+1} = L \cdot (L^n)$

Alternatively:

- $L^n = \{w^n \mid w \in L\}$

Examples

- $\{a, b\}^0 = \{\epsilon\}$
- $\{a, b\}^1 = \{a, b\}$
- $\{a, b\}^2 = \{aa, ab, ba, bb\}$

The Kleene star

- $L^\star = \{w^n \mid w \in L \land n \geq 0\}$

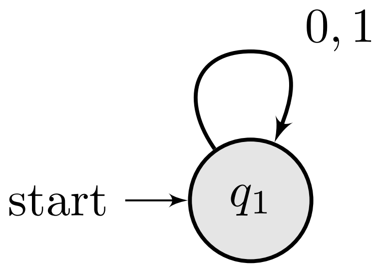
Examples

- $\{a\}^\star = \{w \mid \text{words that only contain } a\}$
- $\{a, b\}^\star = \{a, b\}^0 \cup \{a, b\}^1 \cup \{a, b\}^2 \cup \cdots \cup \{a, b\}^n$

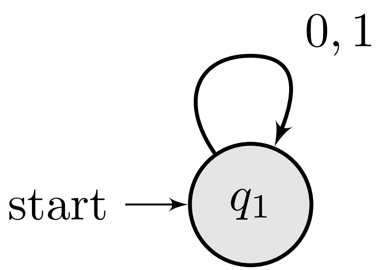
The nil automaton

$$L(\mathbf{nil}_\Sigma) = \emptyset$$

The nil automaton



The nil automaton



- Note the absence of accepted states

```python
def make_nil(alphabet):
  Q1 = "q_1"
  def transition(q, a):
    return q
  return DFA([Q1], alphabet, transition, Q1, lambda x: False)
```

The empty automaton

$$L(\mathbf{empty}_\Sigma) = \{\epsilon\}$$

The empty automaton

> Build an automaton that *only* accepts the empty string ϵ. You can imagine it to be akin to the zero of finite automatons.

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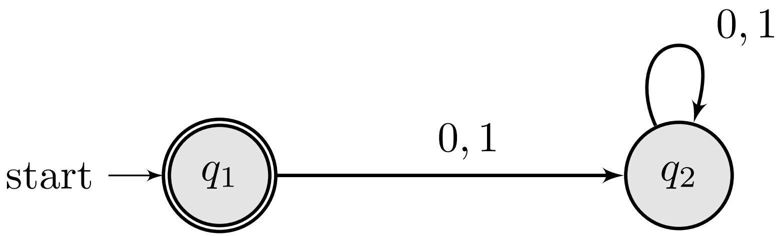
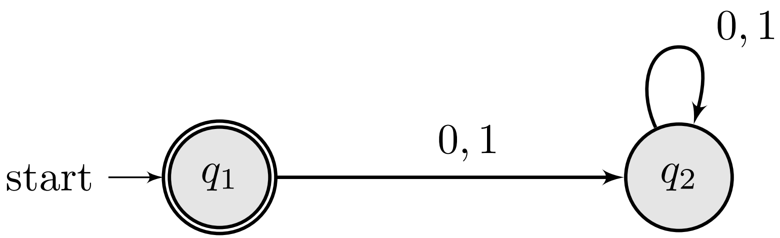



```
def make_empty(alphabet):
  return DFA(
    states = ["q_1", "q_2"],
    alphabet = alphabet,
    transition_func = lambda q, a: "q_2",
    start_state = "q_1",
    accepted_states = lambda x: x == "q_1")
```

The empty automaton

We define function $\mathbf{zero}$ that takes an alphabet Σ as input and outputs an automation that only accepts the empty string whose alphabet is Σ.



$$\mathbf{empty}_\Sigma = (\{q_1, q_2\}, \Sigma, \delta, q_1, \{q_1\})$$

where

$$\delta(q, i) = q_2$$

The character-automaton

$$L(\mathbf{char}(a)) = \{a\}$$

The character automaton

> Given some character `c`, build an automaton that only accepts string `[c]`. This is akin to the numeral 1.

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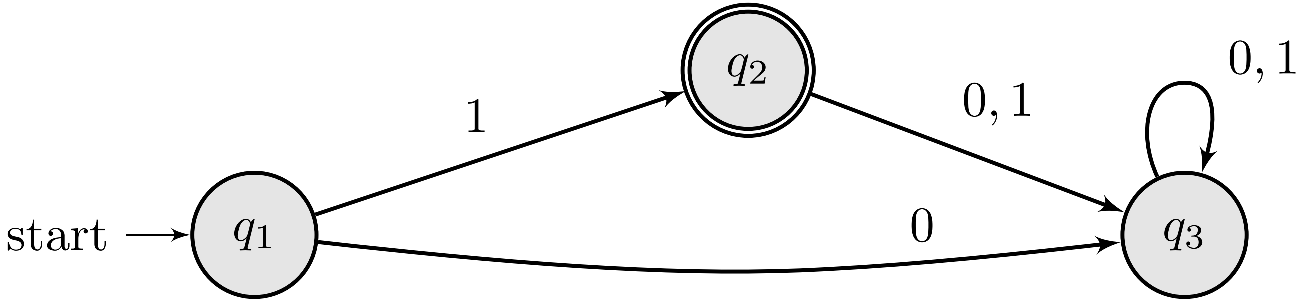
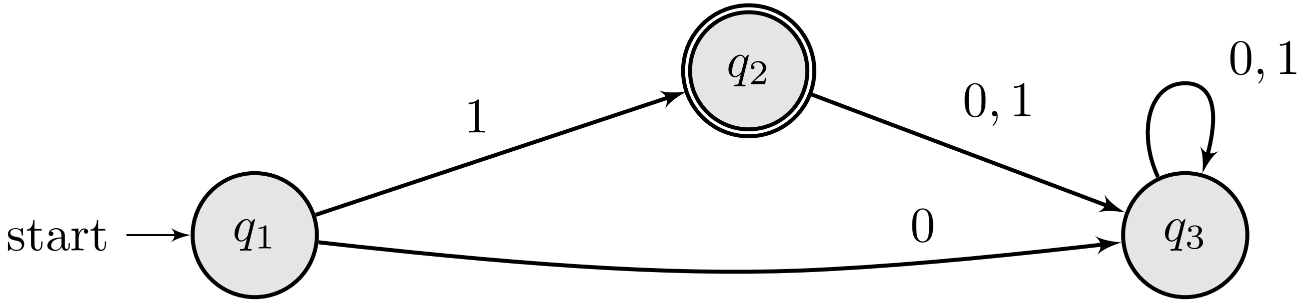


The character automaton



Implementation

```
def make_char(alphabet, char):
  return DFA(
    states=["q_1", "q_2", "q_3"],
    alphabet=alphabet,
    transition_func=lambda q, a: "q_2" if q == "q_1" and a == char else "q_3",
    start_state="q_1",
    accepted_states = lambda x: x == "q_2")
```

The character automaton

We define a function char that takes an alphabet Σ, a function eq that tests if two elements of Σ are equal, and a character $c \in \Sigma$ as input and outputs an automation that only accepts the string `[c]` and whose alphabet is Σ.

$$\mathrm{char}_\Sigma(c) = (\{q_1, q_2, q_3\}, \Sigma, \delta, q_1, \{q_2\})$$

where

2. $c \in \Sigma$
3. $\delta(q_1, c) = q_2$ (Note: This says that the arguments must be exactly q_1 and c.)
   $\delta(q, i) = q_3$ (otherwise)

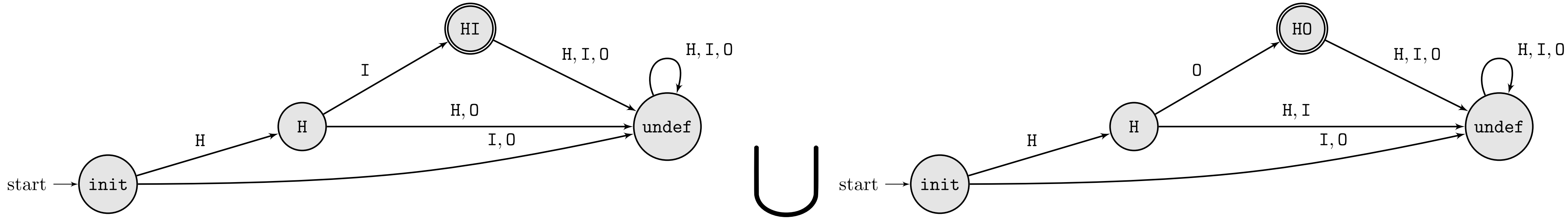
The union automaton

$$L(\text{union}(M_1, M_2)) = L(M_1) \cup L(M_2)$$

The union automaton

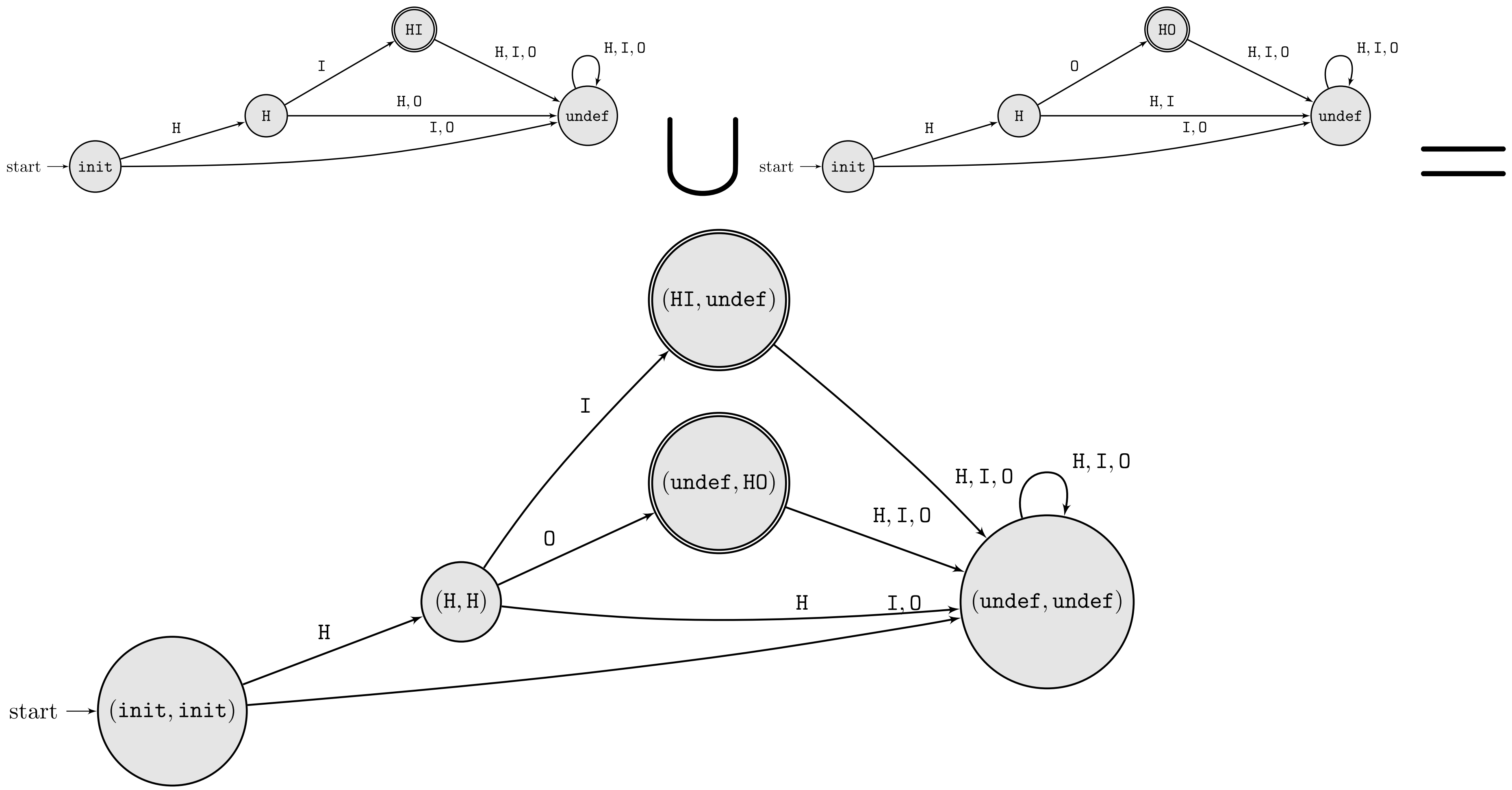
> Formally, the union of two automatons is defined as the union of the recognized languages.

Definition. Say M_1 recognizes A_1 and M_2 recognizes A_2, then $M_1 \cup M_2$ accepts $A_1 \cup A_2$
.



$= ?$

Visually



start
init
H
HI
undef
I
H, I, O
H, O
I, O
∪
HO
O
H, I
=
(init, init)
(H, H)
(HI, undef)
(undef, HO)
(undef, undef)

Implementing the union of DFAs

```python
def union(dfa1, dfa2):
  def transition(q, a):
    return (dfa1.transition_func(q[0], a), dfa2.transition_func(q[1], a))

  def is_final(q):
    return dfa1.accepted_states(q[0]) or dfa2.accepted_states(q[1])

  return DFA(
    states = set(product(dfa1.states, dfa2.states)),
    alphabet = set(dfa1.alphabet).union(dfa2.alphabet),
    transition_func = transition,
    start_state = (dfa1.start_state, dfa2.start_state),
    accepted_states = is_final
  )
```

Formalizing the union

The union operation is defined as $\mathbf{union}(M_1, M_2) = (Q_{1,2}, \Gamma_1, \delta_{1,2}, q_{1,2}, F_{1,2})$ where

- $M_1 = (Q_1, \Gamma_1, \delta_1, q_1, F_1)$
- $M_2 = (Q_2, \Gamma_2, \delta_2, q_1', F_2)$
- **States:** $Q_{1,2} = Q_1 \times Q_2$
- **Alphabet:** $\Gamma_1 = \Gamma_2$
- **Transition:** $\delta_{1,2}(q, a) = (\delta_1(q|_1, a), \delta_2(q|_2, a))$
- **Initial:** $q_{1,2} = (q_1, q_1')$
- **Final:** $F_{1,2} = \{q \mid q|_1 \in F_1 \vee q|_2 \in F_2\}$

Let notation $q|_1 = x$ be defined when $q = (x, y)$. Let notation $q|_2 = y$ be defined when $q = (x, y)$.

The concatenation automaton

$$L(\mathbf{concat}(M_1, M_2)) = L(M_1) \cdot L(M_2)$$

Building a concatenation automation is non-trivial!

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Idea: new formalism!

(To be continued...)

Exercise

Draw an automaton that **recognizes** $\{w \mid w \text{ starts with 10 or ends with 01}\}$.

Exercise

Draw an automaton that **recognizes** $\{w \mid w \text{ starts with } 10 \text{ or ends with } 01\}$.

Idea: separate into two languages and then apply the union automaton

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Draw an automaton that **recognizes** $\{w \mid w \text{ starts with } 10\}$.

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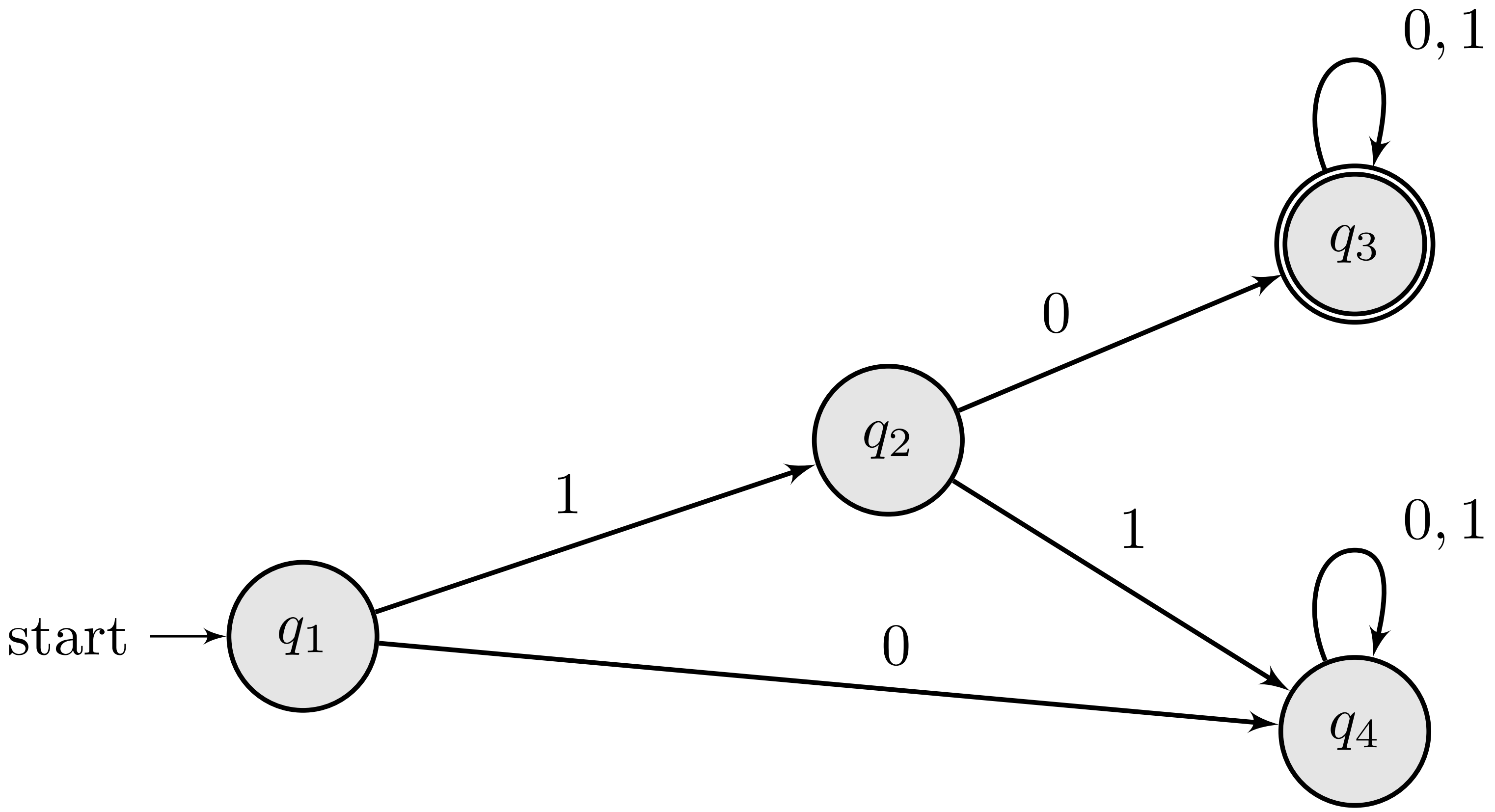

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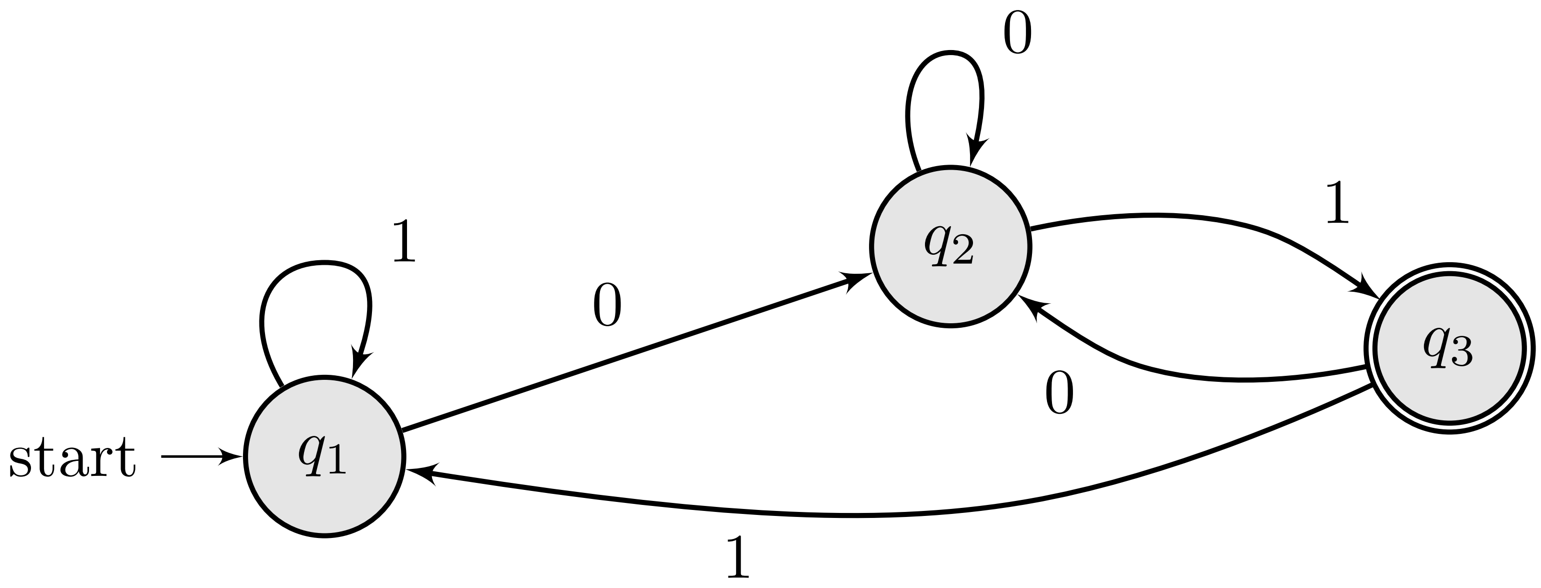

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